The Maximum Permanent of a Stochastic Matrix of Bounded Rank

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Abstract

Let A be a stochastic $n \times n$ matrix with $\operatorname{rank} A \le k$, where $1 \le k \le n$. Write $n = qk + s$, where $0 \le s < k$. The author conjectured in 2018 that

$$\operatorname{per} A \le \left(\frac{q!}{q^q}\right)^{k-s} \left(\frac{(q+1)!}{(q+1)^{q+1}}\right)^{s},$$

with equality if and only if

$$A = P\left(J_q^{\oplus(k-s)} \oplus J_{q+1}^{\oplus s}\right) Q,$$

where P, Q are permutation matrices and J_t is the $t \times t$ matrix with every entry $1/t$. We prove this conjecture in full.

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1 Introduction and main theorem

For an $n \times n$ matrix $A = (a_{ij})$, its permanent is

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{1.1}$$

A nonnegative matrix is **row stochastic**, **column stochastic**, or **doubly stochastic** when all its row sums, all its column sums, or both, respectively, are one. Write Ω_n for the set of doubly stochastic $n \times n$ matrices.

Throughout, rank denotes ordinary matrix rank.

The elementary bound $\operatorname{per} A \le 1$ for a row- or column-stochastic matrix is attained exactly by permutation matrices, all of which have full rank.

For a positive integer t, let $\mathbf{1}_t \in \mathbb{R}^t$ denote the column vector whose entries are all one, and set

$$J_t = \frac{1}{t}\mathbf{1}_t\mathbf{1}_t^{\mathsf{T}}.$$

When its length is clear, we write simply $\mathbf{1}$.

Given $1 \le k \le n$, write

$$n = qk + s, \qquad 0 \le s < k, \tag{1.2}$$

and let $\rho_{n,k}$ denote the balanced k-part partition consisting of $k-s$ parts q and s parts $q+1$. For a partition $\rho = (t_1, \dots, t_k)$, put

$$J_\rho = J_{t_1} \oplus \cdots \oplus J_{t_k}, \qquad C_{n,k} = \operatorname{per} J_{\rho_{n,k}} = \prod_{t \in \rho_{n,k}} \frac{t!}{t^t}. \tag{1.3}$$

The author conjectured in 2018 [1] that every nonnegative row- or column-stochastic $n \times n$ matrix A with $\operatorname{rank} A \leq k$ satisfies

$$\operatorname{per} A \leq C_{n,k} = \left(\frac{q!}{q^q}\right)^{k-s} \left(\frac{(q+1)!}{(q+1)^{q+1}}\right)^s. \tag{1.4}$$

The author also conjectured that equality holds exactly when $A = PJ_{\rho_{n,k}}Q$ for permutation matrices P, Q. We prove both claims for every order and rank ceiling.

Theorem 1.1. *Let $1 \leq k \leq n$, and let $A \geq 0$ be an $n \times n$ matrix with* $\operatorname{rank} A \leq k$.

1. *If A is row stochastic or column stochastic, then*

$$\boxed{\operatorname{per} A \leq C_{n,k}.} \tag{1.5}$$

Equality holds exactly when

$$A = PJ_{\rho_{n,k}}Q \tag{1.6}$$

for permutation matrices P, Q.

2. *For an arbitrary nonnegative A, let r_i and c_j be its row and column sums, and put*

$$R(A) = \prod_{i=1}^{n} r_i, \qquad C(A) = \prod_{j=1}^{n} c_j.$$

Then

$$\boxed{\operatorname{per} A \leq C_{n,k} \min\{R(A), C(A)\}.} \tag{1.7}$$

For completeness, the equality cases in (1.7) are as follows. A zero row or column always gives equality. Otherwise all line sums are positive; define

$$A^r = \operatorname{diag}(r_1^{-1}, \dots, r_n^{-1})A, \qquad A^c = A \operatorname{diag}(c_1^{-1}, \dots, c_n^{-1}). \tag{1.8}$$

If $R(A) < C(A)$, equality holds exactly when $A^r = PJ_{\rho_{n,k}}Q$. If $C(A) < R(A)$, it holds exactly when $A^c = PJ_{\rho_{n,k}}Q$. If $R(A) = C(A) > 0$, equality holds exactly when both normalizations in (1.8) have the displayed balanced-block form; equivalently, it is enough that either one does.

Remark. Parts (1) and (2) are equivalent: Lemma 2.1 proves the equivalence of the inequalities, and the same normalizations identify their equality cases. We state both because the proof proceeds by simultaneous induction on the stochastic and homogeneous formulations.

Every matrix in (1.6) has rank exactly k, so the rank ceiling is sharp.

In the singular specialization $k = n-1$, the balanced partition is $(2, 1, \dots, 1)$, and Theorem 1.1 gives the following particularly simple consequence conjectured in [1].

Corollary 1.2. *Let $n \geq 2$. If A is a singular row- or column-stochastic $n \times n$ matrix, then*

$$\operatorname{per} A \leq \frac{1}{2}. \tag{1.9}$$

Equality holds exactly when $A = P(J_2 \oplus I_{n-2})Q$ for permutation matrices P, Q.

The entire upper half of the rank range also has a simple closed form.

Corollary 1.3. *Let $n/2 < k \leq n$. If A is a row- or column-stochastic $n \times n$ matrix with* $\operatorname{rank} A \leq k$, *then*

$$\operatorname{per} A \leq \frac{1}{2^{n-k}}. \tag{1.10}$$

Equality holds exactly when

$$A = P(J_2^{\oplus(n-k)} \oplus I_{2k-n})Q$$

for permutation matrices P, Q. For $n \geq 3$, Corollary 1.2 is the case $k = n - 1$.

When the balanced partition has equal parts, the general constant likewise simplifies.

Corollary 1.4. *Let $n = qk$, where $q, k \geq 1$. If A is a row- or column-stochastic $n \times n$ matrix with* $\operatorname{rank} A \leq k$, *then*

$$\operatorname{per} A \leq \left(\frac{q!}{q^q}\right)^k. \tag{1.11}$$

Equality holds exactly when $A = P(J_q^{\oplus k})Q$ for permutation matrices P, Q.

Remark. Each of these corollaries has a corresponding form for arbitrary nonnegative matrices: multiply the right-hand side of its displayed inequality by $\min\{R(A), C(A)\}$. The equality cases, including the zero-line cases, are precisely those specified after Theorem 1.1 through the normalizations in (1.8).

Rank-sensitive upper bounds for permanents of doubly stochastic matrices go back at least to Marcus and Minc [2, Corollaries 3.1 and 3.3]. They proved that, for $A \in \Omega_n$,

$$\operatorname{per} A \leq \sqrt{\frac{\operatorname{rank} A}{n}}, \tag{1.12}$$

and, if A is normal, the stronger bound

$$\operatorname{per} A \leq \frac{\operatorname{rank} A}{n}. \tag{1.13}$$

Equality in (1.12) requires a permutation matrix. The normal inequality is strict except for a permutation matrix and the exceptional matrix J_2. These are earlier permanent bounds expressed directly in terms of ordinary rank, but they do not determine the maximum at a prescribed rank or its equality cases. For example, at the singular ceiling $k = n - 1$, the general Marcus–Minc bound (1.12) gives $\sqrt{(n-1)/n}$, while their stronger bound (1.13) for normal matrices gives $(n-1)/n$. For $n \geq 3$, both values exceed the sharp value $1/2$ from Corollary 1.2.

The proof has three main ingredients. First, a rank-safe scaling reduction shows that it is enough to consider doubly stochastic matrices. Second, Tverberg's theorem and a reciprocal linear-programming duality imply that a doubly stochastic matrix of rank at most k has a row whose entries are all at most $1/\lceil n/k \rceil$. Third, expansion along this row transfers the problem to order $n - 1$. Sections 2–4 establish these ingredients and the sharp value. Section 5 proves the equality classification.

2 Normalization and doubly stochastic reduction

2.1 Homogeneous normalization

We begin with two elementary observations that will also organize the induction.

Lemma 2.1. *Fix n, k and a constant $\Gamma > 0$. The following assertions are equivalent.*

1. *Every nonnegative row- or column-stochastic $n \times n$ matrix A with* $\operatorname{rank} A \le k$ *satisfies* $\operatorname{per} A \le \Gamma$.
2. *Every nonnegative $n \times n$ matrix A with* $\operatorname{rank} A \le k$ *satisfies*

$$\operatorname{per} A \le \Gamma \min\{R(A), C(A)\}. \tag{2.1}$$

Proof. Assume (1). If some row sum or column sum is zero, nonnegativity makes the corresponding row or column zero, so both sides of (2.1) vanish. Otherwise all line sums are positive. Left normalization gives

$$\operatorname{per} A = R(A) \operatorname{per} A^r, \qquad \operatorname{rank} A^r = \operatorname{rank} A. \tag{2.2}$$

Similarly, right normalization gives

$$\operatorname{per} A = C(A) \operatorname{per} A^c. \tag{2.3}$$

Applying (1) to A^r and A^c yields (2.1), proving (2).

Conversely, assume (2). If A is row stochastic, then $R(A) = 1$; if it is column stochastic, then $C(A) = 1$. In either case $\min\{R(A), C(A)\} \le 1$, so (2.1) gives $\operatorname{per} A \le \Gamma$. This proves (1). ☐

The same identities prove the homogeneous equality statement in Theorem 1.1 once the stochastic equality cases are known.

Lemma 2.2. *Every nonnegative row- or column-stochastic $n \times n$ matrix A satisfies*

$$\operatorname{per} A \le 1, \tag{2.4}$$

with equality exactly at the permutation matrices.

Proof. For a row-stochastic matrix,

$$1 = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \right) = \sum_{f:[n]\to[n]} \prod_{i=1}^{n} a_{i,f(i)}. \tag{2.5}$$

The permanent is the subsum indexed by bijections, proving the bound. If equality holds, every noninjective term is zero. If two different rows had positive entries in a common column, choosing those entries and arbitrary positive entries in the remaining rows would give a positive noninjective term. Thus the nonempty row supports are pairwise disjoint subsets of an n-element set. Each is a singleton, and row stochasticity makes A a permutation matrix. The column-stochastic case follows by transposition. ☐

2.2 A rank-safe Sinkhorn reduction

We use the standard total-support form of the Sinkhorn–Knopp scaling theorem [3]: a nonzero square nonnegative matrix can be scaled by positive diagonal matrices to a doubly stochastic matrix if every positive entry lies on a positive diagonal.

For a nonnegative matrix $M = (m_{ij})$ and $\sigma \in S_n$, call

$$\prod_{i=1}^{n} m_{i,\sigma(i)}$$

the **permanent monomial** indexed by σ. We call the monomial positive when its value is positive, and in that case call σ a **positive permutation** for M. Equivalently, σ is positive for M exactly when $m_{i,\sigma(i)} > 0$ for every i.

Proposition 2.3. *Let $A \geq 0$ be row stochastic and suppose* $\operatorname{per} A > 0$*. There is a $B \in \Omega_n$ such that*

$$\operatorname{rank} B \leq \operatorname{rank} A, \qquad \operatorname{per} A \leq \operatorname{per} B. \tag{2.6}$$

Furthermore, if A is not doubly stochastic, B can be chosen so that $\operatorname{per} A < \operatorname{per} B$*.*

Proof. Choose a positive permanent monomial, indexed by σ, and permute columns so that the main diagonal of A is positive. We will undo this permutation on both A and the constructed matrix B at the end. On the vertex set $[n]$, form the directed graph having an arc $i \to j$ exactly when $a_{ij} > 0$. An entry $a_{ij} > 0$ belongs to a positive permanent monomial exactly when the arc $i \to j$ lies on a directed cycle. Indeed, a positive permutation decomposes into such cycles. Conversely, a simple directed cycle containing $i \to j$ (with a loop $i \to i$ allowed), together with the positive main-diagonal entries outside that cycle, is a positive permutation.

Call these entries **active**, and let A° retain the active entries and set all other entries to zero. Recall that two vertices belong to the same strongly connected component when each is reachable from the other by a directed path. Mutual reachability is an equivalence relation and hence partitions $[n]$ into components $V_1, \ldots, V_t$. A positive arc $i \to j$ lies on a directed cycle if and only if i and j belong to the same component: a cycle gives paths in both directions, while a path $j \rightsquigarrow i$ closes the arc $i \to j$ to a cycle. Thus A° retains exactly the positive entries whose row and column indices lie in the same component.

Form the directed graph whose vertices are the components and whose arrows record the positive entries of A between distinct components. This component graph is acyclic. Indeed, a directed cycle of components would make all of its vertices mutually reachable and hence part of a single strongly connected component. We may therefore order the components so that every arrow between distinct components points from an earlier component to a later one; this is a topological ordering. Applying this same order to the rows and columns makes A block upper triangular. The matrix A° deletes all entries between distinct components and is therefore block diagonal. Each diagonal block is square because the same set V_h indexes its rows and columns.

Let $A_h = A[V_h, V_h]$ be the h-th diagonal block and put $d_h = \operatorname{rank} A_h$. Choose subsets $R_h, C_h \subseteq V_h$, each of size d_h, such that $\det A_h[R_h, C_h] \neq 0$. Set $R = \bigcup_h R_h$ and $C = \bigcup_h C_h$. In the topological component order, the square submatrix $A[R, C]$ is block upper triangular with diagonal blocks $A_h[R_h, C_h]$. Consequently,

$$\det A[R, C] = \prod_{h=1}^{t} \det A_h[R_h, C_h] \neq 0.$$

Thus A has a nonsingular minor of size $\sum_h d_h$, and hence

$$\operatorname{rank} A^\circ = \sum_{h=1}^{t} d_h \le \operatorname{rank} A. \tag{2.7}$$

Every nonzero permanent monomial of A uses only active entries, so

$$\operatorname{per} A^\circ = \operatorname{per} A. \tag{2.8}$$

The matrix A° has total support. By Sinkhorn–Knopp, there are positive diagonal matrices D, E such that

$$B = DA^\circ E \in \Omega_n. \tag{2.9}$$

It remains to check the direction of the permanent inequality. For a nonnegative $n \times n$ matrix M and $x = (x_1, \dots, x_n) \in \mathbb{R}^n_{>0}$, define

$$p_M(x) = \prod_{i=1}^{n} (Mx)_i, \qquad \operatorname{Cap}(p_M) = \inf_{x \in \mathbb{R}^n_{>0}} \frac{p_M(x)}{x_1 \cdots x_n}. \tag{2.10}$$

If B is doubly stochastic, weighted AM–GM gives

$$\prod_i (Bx)_i \ge \prod_i \prod_j x_j^{b_{ij}} = \prod_j x_j,$$

and equality holds at $x = \mathbf{1}$. Thus $\operatorname{Cap}(p_B) = 1$. If

$$\gamma = (\det D)(\det E),$$

then a change of variables in (2.10) gives

$$1 = \operatorname{Cap}(p_B) = \gamma \operatorname{Cap}(p_{A^\circ}). \tag{2.11}$$

Every row sum of A° is at most the corresponding row sum of A, which is one. Therefore

$$\operatorname{Cap}(p_{A^\circ}) \le p_{A^\circ}(\mathbf{1}) \le 1, \tag{2.12}$$

so $\gamma \ge 1$. Positive diagonal scaling and (2.7)–(2.9) now give

$$\operatorname{per} B = \gamma \operatorname{per} A^\circ \ge \operatorname{per} A, \qquad \operatorname{rank} B = \operatorname{rank} A^\circ \le \operatorname{rank} A. \tag{2.13}$$

This proves the asserted domination. It remains to show that, when the original A is not doubly stochastic, the constructed matrix B satisfies $\operatorname{per} A < \operatorname{per} B$. We prove the contrapositive: suppose $\operatorname{per} A = \operatorname{per} B$. Since the permanent is positive, $\gamma = 1$. Equations (2.11)–(2.12) force every row sum of A° to be one. Thus $A = A^\circ$. Consider the convex function

$$F(z) = \sum_{i=1}^{n} \log\left(\sum_{j=1}^{n} a_{ij} e^{z_j}\right) - \sum_{j=1}^{n} z_j. \tag{2.14}$$

Putting $x_j = e^{z_j}$ in (2.10), and using $A = A^\circ$ and $\gamma = 1$, gives $\inf F = \log \operatorname{Cap}(p_A) = -\log \gamma = 0$. Since $F(0) = 0$, zero minimizes F, and

$$0 = \frac{\partial F}{\partial z_j}(0) = \sum_{i=1}^{n} a_{ij} - 1 \tag{2.15}$$

for every j. The column sums of A are one. Since A was row stochastic, it is therefore doubly stochastic, as required. Undoing the initial column permutation preserves row stochasticity, the property of being doubly stochastic, permanent, and rank, and therefore gives the asserted matrix for the original A. Thus, when the original A is not doubly stochastic, the inequality in (2.6) is strict. $\square$

Consequently, a sharp inequality and equality classification on Ω_n automatically gives the corresponding row- and column-stochastic result without increasing ordinary rank.

3 Tverberg theory and a diffuse row

3.1 A diagonal-rank theorem

We use the classical affine Tverberg theorem [4]: any

$$(r-1)(d+1)+1 \tag{3.1}$$

points, not necessarily distinct, in an affine space of dimension at most d can be partitioned into r nonempty sets whose convex hulls have a common point. If more points are present, the extras may be distributed among the parts.

Theorem 3.1. *Let $P = (p_{ij}) \geq 0$ be an $n \times n$ row-stochastic matrix with* $\operatorname{rank} P \leq k$*, and put*

$$m = \left\lceil \frac{n}{k} \right\rceil .$$

Then

$$\boxed{\min_i p_{ii} \leq \frac{1}{m}.} \tag{3.2}$$

The constant is sharp.

Proof. Write p_i for row i of P. The rows lie in the rowspace of P, and all lie in the affine hyperplane with coordinate sum one. Their affine dimension is therefore at most $k-1$. Since

$$(m-1)k < n, \qquad \text{hence} \qquad n \geq (m-1)k+1, \tag{3.3}$$

Tverberg's theorem gives a partition

$$[n] = S_1 \sqcup \cdots \sqcup S_m \tag{3.4}$$

and a probability vector x belonging to every $\operatorname{conv}\{p_i : i \in S_t\}$. Choose coefficients $\lambda_i^{(t)} \geq 0$, summing to one over $i \in S_t$, such that

$$x = \sum_{i \in S_t} \lambda_i^{(t)} p_i. \tag{3.5}$$

If every $p_{ii} > 1/m$, nonnegativity would give

$$
\begin{aligned}
x(S_t) &= \sum_{j\in S_t} x_j \\
&= \sum_{i\in S_t} \lambda_i^{(t)} \sum_{j\in S_t} p_{ij} \\
&\geq \sum_{i\in S_t} \lambda_i^{(t)} p_{ii} > \frac{1}{m}.
\end{aligned}
\tag{3.6}
$$

Summing over the partition (3.4) would yield $1 > 1$, a contradiction. This proves (3.2).

For sharpness, take $P = J_{\rho_{n,k}}$. Its rank is k, and its smallest diagonal entry is the reciprocal of its largest block size, namely $1/m$. □

Remark. A row-stochastic matrix has spectral radius at most one. At most $\operatorname{rank} P$ of its eigenvalues, counted with algebraic multiplicity, are nonzero. Taking real parts in the trace identity therefore gives

$$\operatorname{tr} P \leq \operatorname{rank} P \leq k.$$

Consequently $\min_i p_{ii} \leq k/n$. When $k \mid n$, this is already the value $1/\lceil n/k \rceil$ in Theorem 3.1. At the singular endpoint $k = n-1$, the same value $1/2$ also follows from strict diagonal dominance: if every $p_{ii} > 1/2$, then $p_{ii} > \sum_{j\neq i} p_{ij}$ in every row, forcing P to be nonsingular. The genuinely stronger numerical content of Theorem 3.1 is the nondivisible range, where

$$\frac{1}{\lceil n/k \rceil} < \frac{k}{n}.$$

The shared indexing of rows and coordinates is essential in this argument. No column-sum condition is assumed.

3.2 A rounded centroid theorem

We now convert Theorem 3.1 into the geometric statement needed for rows of a doubly stochastic matrix.

Theorem 3.2. *Let $1 \leq k \leq n$, and let $v_1, \dots, v_n$ be listed points in an affine space of dimension at most $k-1$. Put*

$$\bar{v} = \frac{1}{n}\sum_{i=1}^{n} v_i, \qquad m = \left\lceil \frac{n}{k} \right\rceil.$$

There is a convex representation

$$\bar{v} = \sum_{i=1}^{n} \lambda_i v_i, \qquad \lambda_i \geq 0, \qquad \sum_i \lambda_i = 1, \tag{3.7}$$

with at most k nonzero coefficients and

$$\boxed{\max_i \lambda_i \geq \frac{m}{n}.} \tag{3.8}$$

The constant is sharp.

Proof. Translate so that $\bar{v} = 0$, choose coordinates in $\mathbb{R}^d$, where $d \le k-1$, and let V be the matrix with columns v_i. Define

$$W = \operatorname{rowspan}\begin{pmatrix} \mathbf{1}^\mathsf{T} \\ V \end{pmatrix}, \qquad L = W^\perp. \tag{3.9}$$

The polytope of representations of the origin is

$$\mathcal{P} = \{p \ge 0 : \mathbf{1}^\mathsf{T} p = 1,\ Vp = 0\} = \left(\frac{1}{n}\mathbf{1} + L\right) \cap \Delta_{n-1}. \tag{3.10}$$

For each coordinate, set

$$\alpha_i = \max_{p \in \mathcal{P}} p_i, \qquad \mu_i = \max_{q \in W \cap \Delta_{n-1}} q_i. \tag{3.11}$$

We first prove the reciprocal identity

$$\boxed{\alpha_i \mu_i = \frac{1}{n}.} \tag{3.12}$$

Write $p = (\mathbf{1} + y)/n$ in (3.10), and put $s_i = n\alpha_i - 1$. Linear-programming duality [5] gives

$$\begin{aligned} s_i &= \max\{y_i : y \in L,\ y_j \ge -1 \text{ for every } j\} \\ &= \min\{\mathbf{1}^\mathsf{T} z : z \ge 0,\ e_i + z \in W\}. \end{aligned} \tag{3.13}$$

The primal is feasible because $y = 0$, and it is bounded above: since $\mathbf{1} \in W$, every $y \in L$ has coordinate sum zero, so the lower bounds $y_j \ge -1$ imply $y_i \le n-1$. Finite-dimensional strong duality therefore applies, and the displayed dual minimum is attained. Every optimal z has $z_i = 0$: if $z_i > 0$, replacing $e_i + z$ by $(e_i + z)/(1 + z_i)$ and then subtracting e_i gives a nonnegative feasible dual vector with smaller coordinate sum. Consequently

$$q = \frac{e_i + z}{1 + s_i} \in W \cap \Delta_{n-1}, \qquad q_i = \frac{1}{1 + s_i} = \frac{1}{n\alpha_i}. \tag{3.14}$$

This proves $\mu_i \ge 1/(n\alpha_i)$. Conversely, if $p \in \mathcal{P}$ and $q \in W \cap \Delta_{n-1}$, then

$$p^\mathsf{T} q = \frac{1}{n} \tag{3.15}$$

because $p - \mathbf{1}/n \in L = W^\perp$. Nonnegativity gives $p_i q_i \le 1/n$; maximizing both coordinates proves the reverse inequality in (3.12).

Suppose that $\alpha_i < m/n$ for every i. Then (3.12) would give $\mu_i > 1/m$ for every i. For each i, choose $q^{(i)} \in W \cap \Delta_{n-1}$ with $q_i^{(i)} = \mu_i$, and stack these vectors as the rows of a matrix Q. Then

$$Q \ge 0, \qquad Q\mathbf{1} = \mathbf{1}, \qquad \operatorname{rank} Q \le \dim W \le k, \qquad q_{ii} > \frac{1}{m}, \tag{3.16}$$

contradicting Theorem 3.1. Hence some $\alpha_i \ge m/n$. A maximizing point may be chosen at a vertex of $\mathcal{P}$. By the standard basic-feasible-solution support bound [5], it has at most $\dim W \le k$ positive coordinates, proving the support assertion.

For sharpness, take k affinely independent point types with the balanced multiplicities in $\rho_{n,k}$. Every representation of their centroid has fixed total weight t/n on a type of multiplicity t. Thus no coefficient can exceed m/n, while concentrating each type weight on one listed copy attains m/n. □

Remark. Ordinary Carathéodory gives a representation of $\bar{v}$ on at most k listed points; since its coefficients sum to one, one coefficient is at least $1/k$. Theorem 3.2 improves this to

$$\frac{\lceil n/k \rceil}{n} \geq \frac{1}{k},$$

with strict improvement exactly when $k \nmid n$.

3.3 The diffuse-row consequence

Corollary 3.3. *If $A \in \Omega_n$ and* $\operatorname{rank} A \leq k$*, then some row a_i satisfies*

$$\boxed{\max_j a_{ij} \leq \frac{1}{\lceil n/k \rceil}.} \tag{3.17}$$

Proof. As in the proof of Theorem 3.1, the rows of A have affine dimension at most $k-1$. Their centroid is the uniform probability vector

$$u = \frac{1}{n}\mathbf{1}^{\mathsf{T}} \tag{3.18}$$

because the column sums are one. By Theorem 3.2,

$$u = \sum_r \lambda_r a_{i_r}$$

with some $\lambda_{r_0} \geq m/n$, where $m = \lceil n/k \rceil$. For every coordinate j, nonnegativity gives

$$\frac{1}{n} = u_j \geq \lambda_{r_0} a_{i_{r_0} j} \geq \frac{m}{n} a_{i_{r_0} j}, \tag{3.19}$$

and hence $a_{i_{r_0} j} \leq 1/m$. □

4 Released-row induction and the sharp value

For $t \geq 1$, write

$$f(t) = \frac{t!}{t^t}.$$

When $n > k$ and $m = \lceil n/k \rceil$, deleting one element from a largest part of $\rho_{n,k}$ produces $\rho_{n-1,k}$. Therefore

$$\frac{C_{n,k}}{C_{n-1,k}} = \frac{f(m)}{f(m-1)} = \left(1 - \frac{1}{m}\right)^{m-1}. \tag{4.1}$$

The same factor arises from the following sharp probability inequality.

Lemma 4.1. *Let $a = (a_1, \ldots, a_N)$ satisfy*

$$a_j \geq 0, \qquad \sum_j a_j = 1, \qquad a_j \leq \frac{1}{m},$$

where $m \geq 2$ is an integer. Then

$$H(a) := \sum_{j=1}^{N} a_j \prod_{\ell \neq j} (1 - a_\ell) \leq \left(1 - \frac{1}{m}\right)^{m-1}. \tag{4.2}$$

Equality holds exactly when, up to permutation,

$$a = (\underbrace{1/m, \dots, 1/m}_{m}, 0, \dots, 0). \tag{4.3}$$

Proof. The feasible set is nonempty only if $N \geq m$, and it is compact, so H has a maximizer. Fix one. Probabilistically, $H(a)$ is the chance of exactly one success among independent Bernoulli variables with success probabilities a_j. Since the coordinates sum to one and each is at most $1/m$, at least $m \geq 2$ are positive. After relabeling, suppose two of them are a_1, a_2. Put $s = a_1 + a_2$ and $v = a_1 a_2$, and let G_0, G_1 be the probabilities of zero and one success among the Bernoulli variables with probabilities $a_3, \dots, a_N$. Recombining the selected pair gives

$$H = (1 - s + v)G_1 + (s - 2v)G_0 = \big((1 - s)G_1 + sG_0\big) + v(G_1 - 2G_0). \tag{4.4}$$

During a fixed-sum perturbation of the selected pair, the remaining coordinates and s are fixed. Thus the parenthesized term in (4.4) is constant and only v varies.

Since $a_r \leq 1/m < 1$, the probability $G_0 = \prod_{r=3}^{N}(1 - a_r)$ is positive. The mutually exclusive choices for the unique successful remaining variable therefore give

$$\frac{G_1}{G_0} = \sum_{r=3}^{N} \frac{a_r}{1 - a_r} \leq \frac{m}{m-1} \sum_{r=3}^{N} a_r = \frac{m}{m-1}(1 - s) < 2. \tag{4.5}$$

The last inequality follows from $s > 0$ and $m/(m-1) \leq 2$. Hence $G_1 - 2G_0 < 0$, so (4.4) shows that H strictly decreases with $v = a_1 a_2$ when $s = a_1 + a_2$ is fixed.

Put $u = 1/m$. Keeping s and the remaining coordinates fixed, write the selected pair as $(z, s - z)$. Its feasible interval is

$$\max\{0, s - u\} \leq z \leq \min\{u, s\}.$$

The product $z(s - z)$ is strictly concave. If both selected coordinates lie strictly between 0 and u, moving z to a suitable endpoint of this interval strictly decreases their product and therefore strictly increases H. At an endpoint, one coordinate equals 0 when $s \leq u$, or one coordinate equals u when $s \geq u$. It follows that a maximizer has at most one coordinate strictly between 0 and u.

If a maximizer had one such coordinate z and r coordinates equal to u, then

$$z = 1 - \frac{r}{m} = \frac{m - r}{m}.$$

The condition $0 < z < 1/m$ would imply $0 < m - r < 1$, which is impossible because $m - r$ is an integer. Thus every maximizer has exactly m coordinates equal to $1/m$ and all other coordinates zero. Substitution gives the value $(1 - 1/m)^{m-1}$, proving both (4.2) and its equality statement. $\square$

We can now prove the value assertion in Theorem 1.1.

Proof of the inequality in Theorem 1.1. Fix k, and induct simultaneously over $n \geq k$ on the stochastic theorem and its homogeneous form. When $n = k$, $C_{k,k} = 1$, so Lemmas 2.1 and 2.2 give the result.

Let $n > k$, and assume the theorem at order $n-1$ with the same rank ceiling k. Let $A \geq 0$ be row stochastic with $\operatorname{rank} A \leq k$. The case $\operatorname{per} A = 0$ is immediate. Otherwise, Proposition 2.3 shows that it is enough to consider $A \in \Omega_n$. Corollary 3.3 supplies a row $a = (a_1, \dots, a_n)$ with $a_j \leq 1/m$, where $m = \lceil n/k \rceil$. Expanding the permanent along that row gives

$$\operatorname{per} A = \sum_{j=1}^{n} a_j \operatorname{per} A(i \mid j), \tag{4.6}$$

where $A(i \mid j)$ denotes the submatrix obtained by deleting the selected row and column j. Its rank is at most k. For every remaining column $\ell \neq j$, the column sum of this submatrix is $1 - a_\ell$. The order-$(n-1)$ homogeneous induction hypothesis therefore gives

$$\operatorname{per} A(i \mid j) \leq C_{n-1,k} \prod_{\ell \neq j} (1 - a_\ell). \tag{4.7}$$

Combining (4.6), (4.7), Lemma 4.1, and (4.1), we obtain

$$\begin{aligned} \operatorname{per} A &\leq C_{n-1,k} H(a) \\ &\leq C_{n-1,k} \left(1 - \frac{1}{m}\right)^{m-1} \\ &= C_{n,k}. \end{aligned} \tag{4.8}$$

This proves the row-stochastic statement at order n; transposition gives the column-stochastic statement, and Lemma 2.1 then gives the homogeneous statement. The simultaneous induction is complete. □

5 Equality cases

It remains to prove that equality in the stochastic theorem forces the balanced blocks. Proposition 2.3 already shows that a row-stochastic equality matrix must be doubly stochastic. We therefore work in Ω_n.

Theorem 5.1. *If $A \in \Omega_n$, $\operatorname{rank} A \leq k$, and*

$$\operatorname{per} A = C_{n,k}, \tag{5.1}$$

then

$$A = P J_{\rho_{n,k}} Q \tag{5.2}$$

for permutation matrices P, Q.

Proof. Fix k and induct on $n \geq k$, together with the value proof. At $n = k$, equality in Lemma 2.2 makes A a permutation matrix, which is (5.2) because $J_{\rho_{k,k}} = I_k$.

Let $n > k$, put $m = \lceil n/k \rceil$, and choose an index i such that the row

$$a := A_{i,*} = (a_1, \dots, a_n)$$

is supplied by Corollary 3.3. Equality throughout (4.8), together with the equality case of Lemma 4.1, forces this row to be

$$a = \frac{1}{m} \mathbf{1}_S, \qquad |S| = m. \tag{5.3}$$

Here $S \subseteq [n]$, and $\mathbf{1}_S \in \mathbb{R}^n$ is its indicator vector: its ℓ-th coordinate is one for $\ell \in S$ and zero otherwise.

Fix $j \in S$, set $T = S \setminus \{j\}$, and let $M = A(i \mid j)$. Since $a_j > 0$, equality in the sum (4.6)–(4.7) forces

$$\operatorname{per} M = C_{n-1,k} \prod_{\ell \neq j} (1 - a_\ell). \tag{5.4}$$

The columns of M indexed by T have sum $(m-1)/m$, and all its other columns have sum one. Scale the T-columns by $m/(m-1)$, and call the resulting matrix D. Then D is column stochastic,

$$\operatorname{rank} D \leq k, \qquad \operatorname{per} D = C_{n-1,k}. \tag{5.5}$$

By the transpose of Proposition 2.3 and the order-$(n-1)$ value bound, D must be doubly stochastic: otherwise the strict part of the proposition would produce a matrix in Ω_{n-1} of rank at most k and permanent strictly greater than $C_{n-1,k}$. The equality-case induction hypothesis now yields

$$D = P' J_{\rho_{n-1,k}} Q'. \tag{5.6}$$

In particular, D has rank exactly k.

Apply the same T-column scaling to all of A, and then delete only column j. After placing the selected row first, the resulting $n \times (n-1)$ matrix has the form

$$\begin{pmatrix} r \\ D \end{pmatrix}, \qquad r = \frac{1}{m-1} \mathbf{1}_T. \tag{5.7}$$

Here $\mathbf{1}_T \in \mathbb{R}^{n-1}$ is the indicator vector of T in the remaining column coordinate set $[n] \setminus \{j\}$.

Column scaling and deletion cannot increase rank, so the rectangular matrix has rank at most k. Since D has rank k, we must have

$$r \in \operatorname{rowspan} D. \tag{5.8}$$

The rowspace of a direct sum of J-blocks consists exactly of the vectors that are constant on each column block. Hence (5.7)–(5.8) show that T is a union of complete column blocks in (5.6).

Since $(m-1)k \leq n-1 < mk$, every part of $\rho_{n-1,k}$ has size $m-1$ or m, and at least one has size $m-1$. Since $|T| = m-1$, the set T is exactly one $(m-1)$-column block. Let R be its associated row block. Before undoing the scaling, this block is J_{m-1}. Undoing the scaling changes all its entries from $1/(m-1)$ to $1/m$. Thus, in the matrix obtained from A by deleting column j, the rows in R, as well as the selected row i, have row sum $(m-1)/m$, while all other rows have row sum one. Row stochasticity of A forces column j to equal $1/m$ on $R \cup \{i\}$ and zero elsewhere. Together with (5.3), the selected row, column j, and the block $R \times T$ form one J_m block. Every other block in (5.6) is unchanged.

Replacing one $(m-1)$-part of $\rho_{n-1,k}$ by an m-part gives exactly $\rho_{n,k}$. This proves (5.2). $\square$

Conversely, each $PJ_{\rho_{n,k}}Q$ is doubly stochastic, has rank k, and has permanent $C_{n,k}$. Theorem 5.1 and Proposition 2.3 therefore give the exact row-stochastic equality class; transposition gives the column class.

Finally, suppose $A \geq 0$ has positive line sums. Equations (2.2)–(2.3) show that equality in the smaller of the two homogeneous bounds is exactly equality for the corresponding stochastic normalization. If $R(A) = C(A)$, equality in one normalization is automatically equality in the other.

Together with the zero-row and zero-column cases, this is exactly the homogeneous classification stated after Theorem 1.1. The proof of the theorem is complete.

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