A counterexample to the Foregger–Sinkhorn tie-point conjecture

Yair Lavi

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Abstract

The Foregger–Sinkhorn tie-point conjecture asserts that if a nearly decomposable doubly stochastic matrix minimizes the permanent on a face and the permanental cofactor at a prescribed zero is larger than its permanent, then that zero is a tie point. We give a counterexample in dimension eight.

Keywords. Permanent; doubly stochastic matrix; nearly decomposable matrix; tie point; extremal matrix problem.
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1 Introduction

Let Ω_n denote the polytope of n-by-n doubly stochastic matrices. For a set Z of positions, put

$$\Omega_n(Z) = \{A = (a_{ij}) \in \Omega_n : a_{ij} = 0 \text{ for every } (i, j) \in Z\}.$$

A square nonnegative matrix is **partly decomposable** if it contains an s-by-$(n - s)$ zero submatrix for some $1 \leq s < n$, and is **fully indecomposable** otherwise. It is **nearly decomposable** if it is fully indecomposable but replacing any one of its positive entries by zero makes it partly decomposable.

Let S be a nearly decomposable support and let $g \notin S$ be a zero position. Hartfiel [1, p. 224] called g a **tie point** if $(S \cup \{g\}) \setminus \{e\}$ is partly decomposable for every original support position $e \in S$. Equivalently, g fails to be a tie point as soon as $(S \cup \{g\}) \setminus \{e\}$ is fully indecomposable for some $e \in S$.

For a matrix A, write $A(i \mid j)$ for the submatrix obtained by deleting row i and column j.

Minc states the Foregger–Sinkhorn conjecture as Conjecture 41 in his 1987 survey [4, p. 135]; Cheon and Wanless retain the same numbering in their later catalogue [5, p. 333]. Its statement is the following.

Tie-point conjecture. Let A be a nearly decomposable matrix that minimizes the permanent on $\Omega_n(Z)$, and let $(i, j) \in Z$. If $\operatorname{per} A(i \mid j) > \operatorname{per} A$, then (i, j) is a tie point for A.

As Minc records [4, p. 135], Foregger and Sinkhorn [2] proposed this implication after giving counterexamples to an earlier conjecture of his. Cheon and Wanless [5, p. 333] report that Foregger [3] subsequently verified the tie-point implication for a special family they call complexes, described as two special vertices joined by separate paths. The source terminology is not uniform: the catalogue says “complex,” whereas the title of [3] says “multiplexes.” We follow the catalogue terminology and give a counterexample outside its described family.

The exact order-eight statement appears in Theorem 1. Its proof occupies Sections 2–5. The result is not a minimality theorem: the absence of smaller counterexamples has not been proved.

1.1 The counterexample

For an n-by-n zero-one matrix D, put $Z(D) = [n]^2 \setminus \operatorname{supp} D$ and $\Omega(D) = \Omega_n(Z(D))$. Thus $\Omega(D)$ is the closed face of doubly stochastic matrices whose support is contained in D. Consider the support

$$D_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Its row strings are

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11110000|10000010|01000010|00001110|
00100001|00010001|00001001|00000101.
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Theorem 1. *The cubic $7x^3 - 13x^2 + 12x - 4$ has a unique root β satisfying $59/100 < \beta < 3/5$. Let $Z_8 = Z(D_8)$.*

The support D_8 is nearly decomposable, the matrix below has support exactly D_8, and the permanent on the entire closed face $\Omega(D_8) = \Omega_8(Z_8)$ has the unique global minimizer

$$A_\beta = \begin{pmatrix} \beta/2 & \beta/2 & (1-\beta)/2 & (1-\beta)/2 & 0 & 0 & 0 & 0 \\ 1-\beta/2 & 0 & 0 & 0 & 0 & 0 & \beta/2 & 0 \\ 0 & 1-\beta/2 & 0 & 0 & 0 & 0 & \beta/2 & 0 \\ 0 & 0 & 0 & 0 & \beta/2 & \beta/2 & 1-\beta & 0 \\ 0 & 0 & (1+\beta)/2 & 0 & 0 & 0 & 0 & (1-\beta)/2 \\ 0 & 0 & 0 & (1+\beta)/2 & 0 & 0 & 0 & (1-\beta)/2 \\ 0 & 0 & 0 & 0 & 1-\beta/2 & 0 & 0 & \beta/2 \\ 0 & 0 & 0 & 0 & 0 & 1-\beta/2 & 0 & \beta/2 \end{pmatrix}.$$

Its permanent is

$$m_8 = \operatorname{per} A_\beta = \frac{13233\beta^2 - 10268\beta + 2536}{67228} \approx 0.01628247758894465.$$

At the prescribed zero $g = (1, 5) \in Z_8$,

$$\operatorname{per} A_\beta(1 \mid 5) - \operatorname{per} A_\beta = \frac{286 + 1599\beta - 3232\beta^2}{9604} > \frac{2047}{240100} > 0.$$

But g is not a tie point. Consequently the Foregger–Sinkhorn tie-point conjecture is false.

2 The support is nearly decomposable

We use the standard bipartite graph of a zero-one support: row vertices and column vertices are joined at the support positions. A square support is fully indecomposable precisely when this graph is connected and every edge lies in a perfect matching. The second condition is usually called total support; Hartfiel records the relevant total-support facts and single-edge deletion lemma [1, pp. 223–224].

Write ij for the edge joining row i to column j. The complete list of perfect matchings of D_8, written as the selected column in rows $1, \dots, 8$, is

$$17253486, \quad 17263458, \quad 21753486, \quad 21763458, \quad 31278456, \quad 41273856. \tag{2.1}$$

This list is exhaustive by splitting on the four possible choices in row 1; the first two choices give two completions each, and the last two give one completion each. The union of the six matchings is all 19 edges of D_8. Moreover,

$$11, 12, 13, 14, 21, 32, 53, 64, 27, 58, 47, 78, 88, 45, 46 \tag{2.2}$$

is a spanning tree of the 16-vertex bipartite graph. Thus D_8 is connected and has total support, so it is fully indecomposable.

It remains to show that every edge is essential. In the following table, the right-hand edge f has the property that every perfect matching containing f also contains the edge e in the left column.

deleted edge e	edge f left without a perfect matching
11	27
12	37
13	58
14	68
21	12
27	11
32	11
37	12
45	78
46	88
47	13
53	11
58	13
64	11
68	14
75	13
78	45
86	13
88	46

The assertion in every row is checked directly from the exhaustive list (2.1). After deleting e, the distinct edge f remains but lies in no perfect matching. Hence the resulting support does not have total support and, by Hartfiel's lemma [1, p. 224], is partly decomposable. This proves that D_8 is nearly decomposable.

The incidence pattern also separates this support from the quoted complex case. Row 1 and column 8 are each incident with four support edges, while row 4 and column 7 are each incident with three; every other vertex is incident with two. Thus D_8 has four vertices where more than two support edges meet, rather than the two special vertices in the catalogue description of a complex.

3 The complete face and its invariant permanent

The global argument must include every boundary stratum of the closed face. Let $a, b, c, d, z, h \geq 0$ satisfy

$$a + b = c + d =: t, \qquad z + h = 1 - t, \qquad 0 \leq t \leq 1. \tag{3.1}$$

These constraints also ensure that every complementary entry displayed below is nonnegative. Then every matrix in $\Omega(D_8)$, and no other matrix, has the unique form

$$A(a,b,c,d,z,h) = \begin{pmatrix} a & b & z & h & 0 & 0 & 0 & 0 \\ 1-a & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 1-b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & c & d & 1-t & 0 \\ 0 & 0 & 1-z & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 1-h & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 1-c & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 1-d & 0 & d \end{pmatrix}. \tag{3.2}$$

To verify this claim directly, start with the entries a, b, z, h in row 1 and c, d in row 4. The sum equations for column 1 and row 2 force the remaining weights $1-a, a$; those for column 2 and row 3 force $1-b, b$. Likewise, columns 3, 4, 5, 6 together with rows 5, 6, 7, 8 force, respectively, the pairs

$$(1-z, z), \qquad (1-h, h), \qquad (1-c, c), \qquad (1-d, d).$$

If x denotes the remaining entry in row 4 and column 7, the four remaining row- and column-sum equations, for row 1, column 7, row 4, and column 8, are

$$a + b + z + h = 1, \qquad a + b + x = 1, \qquad c + d + x = 1, \qquad c + d + z + h = 1.$$

They give

$$a + b = c + d =: t, \qquad z + h = 1 - t, \qquad x = 1 - t.$$

This proves that (3.1)–(3.2) parameterize the entire closed face, including every boundary stratum.

Regard (3.2) as a weighted bipartite graph with vertices R_i, C_j, and give the edge R_iC_j the corresponding matrix-entry weight. The permanent is the sum of the weights of all perfect matchings. Apart from the edge R_4C_7, the graph consists of the following six paths of length three:

path	edge weights
$R_1 - C_1 - R_2 - C_7$	$a,\ 1-a,\ a$
$R_1 - C_2 - R_3 - C_7$	$b,\ 1-b,\ b$
$R_4 - C_5 - R_7 - C_8$	$c,\ 1-c,\ c$
$R_4 - C_6 - R_8 - C_8$	$d,\ 1-d,\ d$
$R_1 - C_3 - R_5 - C_8$	$z,\ 1-z,\ z$
$R_1 - C_4 - R_6 - C_8$	$h,\ 1-h,\ h.$

Call each pair of these paths having the same endpoints a **bundle**. Thus there are three bundles, with path labels (a, b), (c, d), and (z, h). The six paths meet at R_1, C_7, R_4, C_8; we call these four vertices the **hubs**.

On any one of the listed paths, the row- and column-sum equations make the two endpoint-edge weights equal. We call their common value r the **path parameter**. The three edge weights along that path are then $r, 1-r, r$.

Fix a perfect matching of the full graph and consider its restriction to one bundle. All four internal vertices of the bundle must be covered exactly once. For example, the (a,b)-bundle joins R_1 to C_7, and its internal vertices are C_1, R_2, C_2, R_3. If the matching uses neither hub through this bundle, it must select the two middle edges C_1R_2 and C_2R_3, whose product of weights is $(1-a)(1-b)$. If it uses both hubs through this bundle, there are exactly two possibilities:

$$\{R_1C_1, R_2C_7, C_2R_3\} \quad \text{or} \quad \{C_1R_2, R_1C_2, R_3C_7\},$$

with products of weights $a^2(1-b)$ and $(1-a)b^2$, respectively. A configuration using exactly one hub is impossible: on either length-three path, covering its two internal vertices forces either its middle edge alone or both endpoint edges.

More generally, let a bundle have path parameters r, q, and put $s = r+q$ and $p = rq$. The total contribution of configurations using neither hub is

$$G_s(p) = (1-r)(1-q) = 1 - s + p.$$

The total contribution of the two configurations using both hubs is

$$\begin{aligned} F_s(p) &= r^2(1-q) + q^2(1-r) \\ &= s^2 - (s+2)p. \end{aligned} \tag{3.3}$$

Put

$$u = ab, \qquad v = cd, \qquad w = zh. \tag{3.4}$$

The sums and products of the path parameters in the three bundles are (t,u), (t,v), and $(1-t,w)$, respectively. Recording only which hubs are joined, these bundles give the connections R_1C_7, R_4C_8, and R_1C_8. Together with the single edge R_4C_7, they form the cycle

$$R_1 - C_7 - R_4 - C_8 - R_1.$$

Consequently a perfect matching has only two global matching states.

- If R_4C_7 is unused, the (a,b)- and (c,d)-bundles are in state F, while the (z,h)-bundle is in state G.
- If R_4C_7 is used, it contributes $1-t$; the first two bundles are in state G, and the third is in state F.

Summing the contributions of these two disjoint classes of perfect matchings gives

$$P_8(t,u,v,w) = F_t(u)F_t(v)G_{1-t}(w) + (1-t)G_t(u)G_t(v)F_{1-t}(w). \tag{3.5}$$

For nonnegative r, q with $r+q = s$,

$$0 \le rq \le \frac{s^2}{4}, \tag{3.6}$$

because $(r-q)^2 \ge 0$. Equality at the upper endpoint holds exactly when $r = q = s/2$. It follows that

$$0 \le u, v \le \frac{t^2}{4}, \qquad 0 \le w \le \frac{(1-t)^2}{4}.$$

For each fixed t, these inequalities define a three-dimensional box in (u,v,w)-space, possibly degenerate at $t=0$ or $t=1$. Normalize it by

$$u = U\frac{t^2}{4}, \qquad v = V\frac{t^2}{4}, \qquad w = W\frac{(1-t)^2}{4}, \qquad 0 \le U, V, W \le 1. \tag{3.7}$$

In these normalized coordinates, the six bundle factors in (3.5) are

$$\begin{aligned} F_t(u) &= t^2\left(1 - \frac{t+2}{4}U\right), & G_t(u) &= 1 - t + \frac{t^2}{4}U, \\ F_t(v) &= t^2\left(1 - \frac{t+2}{4}V\right), & G_t(v) &= 1 - t + \frac{t^2}{4}V, \\ F_{1-t}(w) &= (1-t)^2\left(1 - \frac{3-t}{4}W\right), & G_{1-t}(w) &= t + \frac{(1-t)^2}{4}W. \end{aligned}$$

The formula (3.5) is affine in each of u, v, w separately. Successive one-variable interpolation in the three normalized coordinates therefore expresses every value as a convex combination of the eight corner values. Specifically, if $C_{ijk}(t)$ is the value of (3.5) at $(U,V,W) = (i,j,k)$, then

$$P_8(t,u,v,w) = \sum_{(i,j,k)\in\{0,1\}^3} C_{ijk}(t) U^i (1-U)^{1-i} V^j (1-V)^{1-j} W^k (1-W)^{1-k}. \tag{3.8}$$

All eight weights are nonnegative, and their sum is $(U+(1-U))(V+(1-V))(W+(1-W)) = 1$. This conclusion is a special consequence of the multiaffine identity (3.8), not a claim that the permanent is a convex function. A corner coordinate 0 means that the corresponding product is zero; when the prescribed sum is positive, one path parameter is zero and the other equals that sum. A coordinate 1 means that the product is maximal, so the two path parameters are equal. At $t=0$ or $t=1$, normalized coordinates on collapsed intervals may be assigned arbitrarily; equivalently, (3.8) at those endpoints follows by continuity.

4 Unique global minimum on the closed face

We first bound the seven corners other than C_{111}, and then treat C_{111} separately.

Lemma 2. *For every $t \in [0,1]$, $C_{ijk}(t) > 1/50$ whenever $(i,j,k) \ne (1,1,1)$.*

Proof. Since $x \mapsto x^5$ is strictly convex on $[0,1]$, Jensen's inequality gives

$$\frac{t^5 + (1-t)^5}{2} \ge \left(\frac{t + (1-t)}{2}\right)^5 = \frac{1}{32}.$$

Hence

$$C_{000}(t) = t^5 + (1-t)^5 \ge \frac{1}{16} > \frac{1}{50}, \tag{4.1}$$

with equality in the first inequality exactly at $t = 1/2$.

For the corner $C_{100} = C_{010}$, write

$$C_{100}(t) = \underbrace{\frac{t^5(2-t)}{4}}_{X(t)} + \underbrace{\frac{(1-t)^4(2-t)^2}{4}}_{Y(t)}. \tag{4.2}$$

On $[0,1]$, X is increasing and Y is decreasing; indeed,

$$X'(t) = \frac{t^4(5-3t)}{2} \geq 0, \qquad Y'(t) = -\frac{(1-t)^3(2-t)(5-3t)}{2} \leq 0.$$

Both X and Y are nonnegative on $[0,1]$. Three elementary estimates now cover the interval. If $t \leq 1/2$, then

$$Y(t) \geq Y(1/2) = \frac{9}{256} > \frac{1}{50}.$$

If $t \geq 3/5$, then

$$X(t) \geq X(3/5) = \frac{1701}{62500} > \frac{1}{50}.$$

Finally, if $1/2 \leq t \leq 3/5$, then

$$X(t) \geq \frac{3}{256} > \frac{1}{100}, \qquad Y(t) \geq \frac{196}{15625} > \frac{1}{100},$$

and hence $C_{100}(t) > 1/50$. The identity $C_{001}(t) = C_{100}(1-t)$ proves the same bound for the remaining asymmetric corner.

For the two corners $C_{101} = C_{011}$, exact differentiation gives

$$\begin{aligned} C_{101}(t) &= \frac{(2-t)(1+t)(t^2-t+1)(7t^2-7t+2)}{16}, \\ C'_{101}(t) &= -\frac{(2t-1)(3t^2-3t+2)(7t^2-7t-8)}{16}. \end{aligned} \tag{4.3}$$

The quadratic $3t^2 - 3t + 2$ is positive, while $7t^2 - 7t - 8 < 0$, throughout $[0,1]$. Thus the derivative has the sign of $2t-1$, and

$$C_{101}(t) \geq C_{101}(1/2) = \frac{27}{1024} > \frac{1}{50}.$$

The last of these seven corners is

$$\begin{aligned} C_{110}(t) &= \frac{(t-2)^2(7t^4-19t^3+25t^2-16t+4)}{16}, \\ C'_{110}(t) &= \frac{(t-2)(3t-2)(7t-10)(2t^2-3t+2)}{16}. \end{aligned} \tag{4.4}$$

On $[0,1]$, the factors $t-2$ and $7t-10$ are negative, and $2t^2-3t+2$ is positive. The derivative therefore has the sign of $3t-2$. Consequently,

$$C_{110}(t) \geq C_{110}(2/3) = \frac{16}{729} > \frac{1}{50}.$$

We have proved

$$C_{ijk}(t) > \frac{1}{50} \qquad \text{for every } (i,j,k) \neq (1,1,1).$$

This proves the lemma. $\square$

The remaining corner is C_{111}. Here the two path parameters in each bundle are equal:

$$a = b = c = d = \frac{t}{2}, \qquad z = h = \frac{1-t}{2}.$$

Its value is

$$q_8(t) = C_{111}(t) = \frac{(t-2)^2(t+1)(8t^4 - 19t^3 + 25t^2 - 16t + 4)}{64}. \tag{4.5}$$

It crosses below the separating level because

$$q_8(3/5) - \frac{1}{50} = -\frac{577}{156250} < 0. \tag{4.6}$$

Its derivative factors as

$$q_8'(t) = \frac{(t-2)(4t^2 - 3t - 4)}{32}(7t^3 - 13t^2 + 12t - 4). \tag{4.7}$$

On $[0,1]$, both $t-2$ and $4t^2 - 3t - 4$ are negative, so the prefactor in (4.7) is positive. Let

$$g(t) = 7t^3 - 13t^2 + 12t - 4.$$

The derivative $g'(t) = 21t^2 - 26t + 12$ is positive on the real line: its leading coefficient is positive and its discriminant is -332. Moreover,

$$g(59/100) = -\frac{7647}{1000000} < 0, \qquad g(3/5) = \frac{4}{125} > 0. \tag{4.8}$$

Thus g has a unique root $\beta \in (59/100, 3/5)$. Since the prefactor in (4.7) is positive, q_8 strictly decreases up to β and strictly increases after it. Hence $q_8(t) \geq q_8(\beta)$ on $[0,1]$, with equality only at $t = \beta$.

Let $m = q_8(\beta)$. By (4.6),

$$m = q_8(\beta) \leq q_8(3/5) < \frac{1}{50}.$$

For an arbitrary point of the closed face, the multiaffine identity (3.8) expresses its permanent as a convex combination of the eight values $C_{ijk}(t)$. By Lemma 2, seven of these values are strictly larger than $1/50 > m$; the remaining value is $C_{111}(t) = q_8(t) \geq q_8(\beta) = m$. Therefore every matrix in the face has permanent at least m.

The equality conditions are equally explicit. Equality requires zero weight on every one of the seven corners other than C_{111}. Since all eight weights in (3.8) are nonnegative and sum to one, the weight on C_{111} must be one; hence $U = V = W = 1$. Equality also requires $q_8(t) = m$, so the uniqueness of the minimum of q_8 gives $t = \beta$. Finally, maximal product at a fixed pair sum is attained uniquely by equal numbers. Thus

$$a = b = c = d = \frac{\beta}{2}, \qquad z = h = \frac{1-\beta}{2}. \tag{4.9}$$

It follows that the displayed matrix A_β is the unique global minimizer on the entire closed face. Reducing (4.5) modulo $g(t)$ gives

$$m = q_8(\beta) = \frac{13233\beta^2 - 10268\beta + 2536}{67228}, \tag{4.10}$$

as asserted in Theorem 1.

5 The prescribed zero is not a tie point

Let $g = (1,5)$, a zero position of D_8. Direct permanent expansion of the 7-by-7 minor gives

$$\operatorname{per} A_\beta(1 \mid 5) = \frac{\beta(\beta-2)^3(\beta-1)(\beta+1)^2}{64}. \tag{5.1}$$

Subtracting (4.5) and reducing modulo $7\beta^3 - 13\beta^2 + 12\beta - 4$ yields

$$\operatorname{per} A_\beta(1 \mid 5) - \operatorname{per} A_\beta = H(\beta), \qquad H(t) = \frac{286 + 1599t - 3232t^2}{9604}. \tag{5.2}$$

Now $H'(t) = (1599 - 6464t)/9604$, and

$$H'(59/100) = -\frac{55369}{240100} < 0.$$

Since H' is decreasing, H is strictly decreasing throughout the isolating interval. Therefore

$$H(\beta) > H(3/5) = \frac{2047}{240100} > 0. \tag{5.3}$$

It remains to show that g is not a tie point. Add $(1,5)$ and delete the original edge $(4,5)$. The surviving support is

```
11111000|10000010|01000010|00000110|
00100001|00010001|00001001|00000101.
```

It has the five perfect matchings

$$17263458, \quad 21763458, \quad 31278456, \quad 41273856, \quad 51273486. \tag{5.4}$$

Their union is the entire displayed support. The edges

$$11, 12, 13, 14, 15, 21, 32, 53, 64, 75, 27, 58, 47, 88, 46 \tag{5.5}$$

form a spanning tree, so the support is connected. It consequently has total support and is fully indecomposable. Thus deletion of the original edge $(4,5)$ from the support augmented at g does **not** make the support partly decomposable. By Hartfiel's definition, g is not a tie point.

The matrix A_β is nearly decomposable by Section 2, is the global minimizer on the prescribed face by Section 4, and satisfies the strict cofactor inequality by (5.3), while the conjectured conclusion fails. This completes the proof of Theorem 1. □

Acknowledgments

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