

The Maximum of $\text{per}(I - A)$ in Odd Order

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Abstract

Let Ω_n denote the set of $n \times n$ doubly stochastic matrices. Kim and Roush conjectured in 1981 that, for $n = 2k + 1 > 1$,

$$\max_{A \in \Omega_{2k+1}} \text{per}(I - A) = 3 \cdot 2^{k-2}.$$

They proposed the following block construction as an extremizer:

$$A_\star = \frac{1}{2}(J_3 - I_3) \oplus P_2^{\oplus(k-1)}, \quad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here J_3 is the 3×3 all-ones matrix. They did not claim uniqueness. We fully prove their conjecture and classify equality: the maximizers are exactly the simultaneous-permutation conjugates of $A_\star$.

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1 Introduction

For an $n \times n$ matrix $D = (d_{ij})$, its permanent is

$$\text{per } D = \sum_{\sigma \in S_n} \prod_{i=1}^n d_{i, \sigma(i)}.$$

We write Ω_n for the Birkhoff polytope of nonnegative doubly stochastic $n \times n$ matrices. The problem considered here is to maximize $\text{per}(I - A)$ over $A \in \Omega_n$. Although $I - A$ generally has negative off-diagonal entries, stochasticity gives this permanent the positive functional-digraph expansion proved in Proposition 3.1.

Kim and Roush [1] proved that for nonnegative row-stochastic matrices

$$\max_A \text{per}(I - A) = 2^{\lfloor n/2 \rfloor}. \tag{1.1}$$

Their proof established the functional-digraph identity used below. For even $n = 2m$, the row-stochastic maximum is attained by a fixed-point-free involution, which is doubly stochastic, and hence the doubly stochastic maximum is 2^m . For odd $n = 2k + 1$, (1.1) gives the upper bound 2^k . Kim and Roush exhibited the doubly stochastic matrix

$$A_\star = \frac{1}{2}(J_3 - I_3) \oplus P_2^{\oplus(k-1)}, \quad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1.2)$$

and conjectured that its value $3 \cdot 2^{k-2}$ is maximal. They proved the case $n = 3$, where the value is $3/2$ [1, Proposition 5]. Thus the open range in their paper began at $n = 5$.

Minc later added the conjecture to his catalogue as Conjecture 35 [2, p. 133]. In their subsequent survey of Minc's list, Cheon and Wanless [3] reported no progress. Troanca [4] reproduced the exact construction and stated in 2008 that no partial results were known to the author. Chen and Cao [5] subsequently solved a neighboring problem with fixed total matrix mass. In their notation, for row- and doubly substochastic matrices of total mass $s = \sum_{i,j} a_{ij}$, they obtained the exact common maximum

$$2^{e/2} \left[1 + \left(\frac{s-e}{2} \right)^2 \right], \quad (1.3)$$

where e is the largest even integer not exceeding s , whenever n is even or n is odd and $s \leq n-1$. For odd n and $n-1 < s \leq n$, they determined the row-substochastic maximum but left the doubly substochastic case open [5, Theorems 3.1, 3.13, and 4.1]. The endpoint $s = n$ is exactly the odd doubly stochastic problem. They also gave an independent proof of the $n = 3$ value [5, Lemma 4.2].

Our main result settles the conjecture in full and classifies all equality cases.

Theorem 1.1 (Kim–Roush conjecture). *For every integer $k \geq 1$,*

$$\boxed{\max_{A \in \Omega_{2k+1}} \text{per}(I - A) = 3 \cdot 2^{k-2}.} \quad (1.4)$$

The matrix $A_\star$ in (1.2) attains equality. Moreover, equality holds for $A \in \Omega_{2k+1}$ if and only if

$$A = P A_\star P^\top \quad (1.5)$$

for a permutation matrix P . Thus the number of distinct maximizers is $\binom{2k+1}{3}(2k-3)!!$, with the convention $(-1)!! = 1$.

Here is the proof in outline. Independently for each $i \in [n]$, choose $F(i) = j$ with probability a_{ij} . These choices define a random map $F : [n] \rightarrow [n]$. Let m_c be the total product mass of maps whose functional digraphs have only even cycles and exactly c cycles, and put $M = \sum_c m_c$. The Kim–Roush expansion and a layer estimate give

$$\text{per}(I - A) = \sum_{c=1}^k 2^c m_c \leq 2^{k-1}(M + m_k). \quad (1.6)$$

Let Q_3 be the mass of permutation maps with cycle type $(3, 2, \dots, 2)$. The key estimate is the sharp triangle-charge inequality

$$m_k - Q_3 \leq \frac{1}{2}. \quad (1.7)$$

Every map counted by Q_3 has an odd cycle, so $Q_3 \leq 1 - M$. Consequently (1.7) implies $M + m_k \leq 3/2$, which inserted into (1.6) proves the upper bound in (1.4).

The proof of (1.7) has two exact ingredients. First, averaging a matrix with its transpose does not decrease the difference $m_k - Q_3$; the local loss is a sum of three squares. Second, for symmetric

weights, releasing the terminal row of an ordered spanning path on three vertices yields a mass B satisfying $2m_k \leq B$ and

$$B = m_k + 3Q_3 + O,$$

where O is a disjoint residual-map mass. Since $O \leq 1 - m_k - Q_3$, inequality (1.7) follows. The factor 3 is the number of ways to open a directed triangle into a directed path.

Sections 2–5 develop the map expansion and prove the triangle-charge inequality. Section 6 establishes the maximum, and Section 7 classifies equality.

2 Maps, masses, and matchings

Fix $n = 2k + 1$ with $k \geq 1$, and let $A = (a_{ij}) \in \Omega_n$. Write $[n] = \{1, \dots, n\}$. For a map $f : [n] \rightarrow [n]$, define its product mass by

$$\mu_A(f) = \prod_{i=1}^n a_{i,f(i)}. \quad (2.1)$$

The row sums of A are one, and therefore

$$\sum_{f:[n] \rightarrow [n]} \mu_A(f) = \prod_{i=1}^n \sum_{j=1}^n a_{ij} = 1. \quad (2.2)$$

The directed graph of f has the edge $i \rightarrow f(i)$ from every vertex. Each weak component contains one directed cycle. Let $c(f)$ denote the number of these directed cycles.

For $1 \leq c \leq k$, let $\mathcal{E}_c$ be the maps all of whose directed cycles have even length and for which $c(f) = c$. Set

$$m_c = m_c(A) = \sum_{f \in \mathcal{E}_c} \mu_A(f), \quad M = M(A) = \sum_{c=1}^k m_c. \quad (2.3)$$

The families $\mathcal{E}_c$ are disjoint subsets of all maps, so $0 \leq M \leq 1$.

For a finite set U of even cardinality, let $\text{PM}(U)$ denote the set of perfect matchings of U . If $W = (w_{ij})_{i,j \in U}$ is symmetric, write

$$\text{haf } W = \sum_{\nu \in \text{PM}(U)} \prod_{\{r,s\} \in \nu} w_{rs}. \quad (2.4)$$

We use the convention that the hafnian of the empty matrix equals one. Put

$$C(A) = A \circ A^T, \quad C(A)_{ij} = a_{ij}a_{ji}, \quad (2.5)$$

where $\circ$ denotes the entrywise (Hadamard) product. For $U \subseteq [n]$, let $W[U]$ denote the principal submatrix of W indexed by U . Whenever $[n] \setminus S$ has even cardinality, define

$$H_S(A) = \text{haf}(C(A)[[n] \setminus S]). \quad (2.6)$$

Thus $H_S(A)$ is the total weight of mutual-pair matchings outside S .

3 The positive expansion and the top layer

We first give a self-contained proof of the expansion from [1].

Proposition 3.1 (even-cycle expansion). *If A is nonnegative and row-stochastic, then*

$$\text{per}(I - A) = \sum_{\substack{f: [n] \rightarrow [n] \\ \text{every cycle of } f \text{ is even}}} 2^{c(f)} \mu_A(f). \quad (3.1)$$

In particular, for $n = 2k + 1$,

$$\text{per}(I - A) = \sum_{c=1}^k 2^c m_c. \quad (3.2)$$

Proof. For $S \subseteq [n]$, let $\mathfrak{S}(S)$ denote the set of permutations of S . Expand the permanent according to the rows in which the term from $-A$ is selected. This gives

$$\text{per}(I - A) = \sum_{S \subseteq [n]} (-1)^{|S|} \sum_{\pi \in \mathfrak{S}(S)} \prod_{i \in S} a_{i, \pi(i)}, \quad (3.3)$$

where the empty product is one. For a fixed pair (S, π) , row-stochasticity gives

$$\prod_{i \in S} a_{i, \pi(i)} = \sum_{\substack{f: [n] \rightarrow [n] \\ f|_S = \pi}} \mu_A(f). \quad (3.4)$$

After interchanging the finite sums, fix a map f . The subsets S for which $f|_S$ is a permutation are exactly the unions of directed cycles of f . Hence the coefficient of $\mu_A(f)$ is

$$\prod_{\gamma \text{ a cycle of } f} (1 + (-1)^{|\gamma|}). \quad (3.5)$$

This coefficient vanishes if f has an odd cycle and equals $2^{c(f)}$ otherwise. Since every finite map has a cycle, and c even cycles use at least $2c$ vertices, in odd order $1 \leq c \leq k$. This proves both formulas. $\square$

The next reduction isolates the only layer requiring sharp control.

Lemma 3.2 (layer reduction). *For $A \in \Omega_{2k+1}$,*

$$\text{per}(I - A) \leq 2^{k-1} (M + m_k). \quad (3.6)$$

Proof. In (3.2), bound $2^c \leq 2^{k-1}$ for $c \leq k-1$, while retaining the coefficient 2^k of the top layer. Thus

$$\text{per}(I - A) \leq 2^{k-1} (M - m_k) + 2^k m_k = 2^{k-1} (M + m_k).$$

For $k = 1$ the first sum is empty and the displayed relation is an identity. $\square$

Call a map in $\mathcal{E}_k$ a **top map**. Such a map has a particularly rigid form.

Lemma 3.3 (top-layer structure). *One has*

$$m_k = \sum_{v=1}^n (1 - a_{vv}) H_{\{v\}}(A) = \sum_{\substack{v,u,w \in [n] \\ v,u,w \text{ distinct}}} a_{vu} a_{uw} a_{wu} H_{\{v,u,w\}}(A). \quad (3.7)$$

Proof. A map in $\mathcal{E}_k$ has k even cycles on $2k + 1$ vertices. It must therefore have k transpositions covering $2k$ vertices and one vertex v not lying on a directed cycle (hence one nonperiodic vertex). The vertex v maps to some $u \neq v$, while the remaining vertices form a perfect matching of mutual pairs. For fixed v and fixed matching, the sum over the target of v is $\sum_{u \neq v} a_{vu} = 1 - a_{vv}$, giving the first formula.

For the second formula, let w be the mate of the chosen target u . The distinguished part of the map has weight $a_{vu} a_{uw} a_{wu}$, and the remaining mutual pairs contribute $H_{\{v,u,w\}}(A)$. The tail vertex, its target, and the target's mate are uniquely determined by the map. $\square$

4 Triangle charge and transpose symmetrization

Let $\mathcal{Q}_3$ be the set of permutation maps whose cycle type is $(3, 2, \dots, 2)$: one directed 3-cycle and $k - 1$ transpositions. Define

$$Q_3 = Q_3(A) = \sum_{f \in \mathcal{Q}_3} \mu_A(f). \quad (4.1)$$

Fix a three-element set $T = \{i, j, \ell\}$. The vertices outside T must be paired into transpositions, whose total weight is $H_T(A)$, while the directed 3-cycle on T has exactly two orientations. Therefore

$$Q_3(A) = \sum_{|T|=3} (a_{ij} a_{j\ell} a_{\ell i} + a_{i\ell} a_{\ell j} a_{ji}) H_T(A). \quad (4.2)$$

Each map in $\mathcal{Q}_3$ contains an odd cycle, whereas all maps counted by M have only even cycles. These map families are disjoint, so (2.2) gives

$$Q_3 \leq 1 - M. \quad (4.3)$$

We call $m_k - Q_3$ the **triangle charge**. To express it locally, group (3.7) and (4.2) by their distinguished three-element set and define

$$K_T(A) = \sum_{\substack{v,u,w \in T \\ v,u,w \text{ distinct}}} a_{vu} a_{uw} a_{wu} - \frac{1}{3} \sum_{\substack{v,u,w \in T \\ v,u,w \text{ distinct}}} a_{vu} a_{uw} a_{wv}. \quad (4.4)$$

The normalization $1/3$ removes a simple overcount; this correction is needed for identity (4.5). In the first sum, the ordered triple (v, u, w) records a top map's nonperiodic vertex v , its target u , and the mate w of u . In the second sum, $a_{vu} a_{uw} a_{wv}$ is the weight of the directed triangle $v \rightarrow u \rightarrow w \rightarrow v$. Each directed triangle on T occurs three times in that sum, once for each cyclic choice of the first vertex v . Thus the second sum, after multiplication by $1/3$, is exactly the sum of the two orientation weights in (4.2). It follows that

$$m_k(A) - Q_3(A) = \sum_{|T|=3} K_T(A) H_T(A). \quad (4.5)$$

Although writing $T = \{i, j, \ell\}$ in (4.2) requires an arbitrary ordering, the sum of the two orientation weights in (4.2), the quantity $K_T(A)$, and identity (4.5) depend only on the set T .

For nonnegative x, y, z , put

$$\begin{aligned}\Phi(x, y, z) &= xy(x + y) + yz(y + z) + zx(z + x) - 2xyz \\ &= (x + y)(y + z)(z + x) - 4xyz \\ &= x(y - z)^2 + y(z - x)^2 + z(x - y)^2 + 4xyz.\end{aligned}\tag{4.6}$$

In particular, $\Phi(x, y, z) \geq 0$.

Lemma 4.1 (local transpose identity). *Let $X = (A + A^\top)/2$. Orient $T = \{i, j, \ell\}$ cyclically as $i \rightarrow j \rightarrow \ell \rightarrow i$, and write*

$$\begin{aligned}a_{ij} &= x + p, & a_{ji} &= x - p, & a_{j\ell} &= y + q, \\ a_{\ell j} &= y - q, & a_{\ell i} &= z + r, & a_{i\ell} &= z - r.\end{aligned}\tag{4.7}$$

Then

$$\frac{K_T(A) + K_T(A^\top)}{2} = \Phi(x, y, z) - D_T,\tag{4.8}$$

where

$$D_T = y(p + r)^2 + z(p + q)^2 + x(q + r)^2 \geq 0.\tag{4.9}$$

Proof. Substitute the six expressions in (4.7) into (4.4), add the expression obtained after changing (p, q, r) to $(-p, -q, -r)$, and collect the even terms. The terms independent of (p, q, r) give (4.6), and the remaining even terms are the negative of (4.9). $\square$

For the variables attached to T in (4.7), abbreviate $\Phi_T = \Phi(x, y, z)$.

Theorem 4.2 (transpose symmetrization). *For every $A \in \Omega_{2k+1}$ and $X = (A + A^\top)/2$,*

$$m_k(A) - Q_3(A) \leq m_k(X) - Q_3(X).\tag{4.10}$$

Proof. Here the column-sum condition enters the upper-bound argument: because A is doubly stochastic, $A^\top$ is row-stochastic and X is symmetric and doubly stochastic. The quantities m_k and Q_3 are transpose invariant. For m_k , apply the first formula in (3.7) to both A and $A^\top$ and use $C(A) = C(A^\top)$; for Q_3 , transposition exchanges the two orientations in (4.2). Moreover $H_T(A) = H_T(A^\top)$. We may therefore replace $K_T(A)$ in (4.5) by its transpose average.

For an edge $\{r, s\}$ outside T , write

$$x_{rs} = \frac{a_{rs} + a_{sr}}{2}, \quad \kappa_{rs} = \frac{a_{rs} - a_{sr}}{2}.$$

Then

$$a_{rs}a_{sr} = x_{rs}^2 - \kappa_{rs}^2, \quad 0 \leq x_{rs}^2 - \kappa_{rs}^2 \leq x_{rs}^2.\tag{4.11}$$

Expanding the hafnian in (4.5) over perfect matchings and using Lemma 4.1 expresses the charge as a sum of terms

$$(\Phi_T - D_T) \prod_{\{r,s\} \in \nu} (x_{rs}^2 - \kappa_{rs}^2).\tag{4.12}$$

The following sign distinction is essential. If $\Phi_T - D_T \leq 0$, the term in (4.12) is nonpositive and hence at most $\Phi_T \prod_{\{r,s\} \in \nu} x_{rs}^2$. If $\Phi_T - D_T > 0$, first discard D_T and then use (4.11), obtaining the same upper bound. Summing over triples and outside perfect matchings gives

$$m_k(A) - Q_3(A) \leq \sum_{|T|=3} \Phi_T H_T(X) = m_k(X) - Q_3(X),$$

where the last equality is (4.5) specialized to the symmetric matrix X . $\square$

Remark 4.3 (why the transpose average is necessary). The unaveraged local inequality suggested by (4.8) is false. For example, on a cyclically oriented triple take

$$(a_{12}, a_{21}, a_{13}, a_{31}, a_{23}, a_{32}) = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right). \quad (4.13)$$

The ungrouped comparison fails by exactly $1/8$. These six entries have row and column subsums at most one and extend to a doubly stochastic matrix. The odd-in-skew remainder cancels only after pairing A with A^T . Thus the transpose invariance used in Theorem 4.2 is a structural part of the proof, not a cosmetic symmetrization.

5 The symmetric charge inequality

We now prove the sharp estimate for a symmetric doubly stochastic matrix $X = (x_{ij})$ of order $2k + 1$. All weights below are entries of X .

For distinct vertices v, u, w and a perfect matching ν of the other $2k - 2$ vertices, define the released-path weight

$$h(v, u, w; \nu) = x_{vu} x_{uw} \prod_{\{a,b\} \in \nu} x_{ab}^2. \quad (5.1)$$

It fixes the choices

$$v \mapsto u, \quad u \mapsto w, \quad a \longleftrightarrow b \quad (\{a, b\} \in \nu), \quad (5.2)$$

and releases the terminal row w . Indeed, summing $w \mapsto t$ over all t multiplies (5.1) by $\sum_t x_{wt} = 1$. Let B be the total released-path mass:

$$B = \sum_{\substack{v, u, w \text{ distinct} \\ \nu \in \text{PM}([n] \setminus \{v, u, w\})}} h(v, u, w; \nu). \quad (5.3)$$

Lemma 5.1 (released mass dominates the top layer). *One has*

$$2m_k(X) \leq B. \quad (5.4)$$

Proof. Pair the paths (v, u, w) and (w, u, v) while fixing the center u and the outside matching ν . Put

$$a = x_{vu}, \quad b = x_{uw}, \quad H = \prod_{\{r,s\} \in \nu} x_{rs}^2.$$

Their combined released mass is $2abH$. Closing their terminal rows back to u produces two top maps of total mass

$$ab^2H + a^2bH = ab(a+b)H.$$

Since the center row is stochastic,

$$a + b \leq \sum_{z \neq u} x_{uz} = 1 - x_{uu} \leq 1.$$

Thus twice the mass of the two top completions is at most their released mass. Every top map has a unique tail vertex v , tail target u , mate w of u , and outside matching ν , so summing proves (5.4). Equivalently, the slack is the manifestly nonnegative sum

$$B - 2m_k = \sum_{u, \{v, w\}, \nu} 2x_{vu}x_{uw}(1 - x_{vu} - x_{uw}) \prod_{\{a, b\} \in \nu} x_{ab}^2. \quad (5.5)$$

Here the sum is over centers u , unordered endpoint pairs $\{v, w\}$ disjoint from u , and matchings $\nu \in \text{PM}([n] \setminus \{u, v, w\})$. The right-hand side is nonnegative. $\square$

We next classify the maps obtained when the released row is completed.

Lemma 5.2 (completion identity). *There is a nonnegative map mass O , disjoint from the masses m_k and Q_3 , such that*

$$B = m_k + 3Q_3 + O \quad (5.6)$$

and

$$O \leq 1 - m_k - Q_3. \quad (5.7)$$

Proof. In a released configuration (5.2), classify the completion by the choice $w \mapsto t$.

If $t = u$, the terminal row closes the mutual pair $u \leftrightarrow w$ and produces a top map. Its representation is unique, so every top map occurs once.

If $t = v$, the path closes into the directed triangle $v \rightarrow u \rightarrow w \rightarrow v$. Every map counted by Q_3 occurs three times, once for each cyclic choice of where its directed triangle is opened.

Finally, suppose $t \notin \{u, v\}$, allowing either $t = w$ or t in an outside pair. Call the resulting map residual. Its mutual pairs are exactly the pairs of ν : none of $\{v, u\}$, $\{u, w\}$, or $\{w, t\}$ is mutual. After removing those pairs, the three remaining vertices contain the unique directed path $v \rightarrow u \rightarrow w$; the vertex v is its unique vertex of in-degree zero, and w either loops or maps out to a removed pair. Hence v, u, w , and ν are uniquely reconstructed from the residual map. Every residual map therefore occurs once.

These three classes are disjoint: top maps have exactly one nonperiodic vertex, Q_3 maps are permutations with a directed odd cycle, and residual maps have at least two nonperiodic vertices. If O is the total product mass of the residual maps, the multiplicities are exactly 1, 3, 1, proving (5.6). All three classes lie inside the map space of total mass one in (2.2), which proves (5.7). $\square$

Theorem 5.3 (symmetric triangle charge). *For every symmetric $X \in \Omega_{2k+1}$,*

$$m_k(X) - Q_3(X) \leq \frac{1}{2}. \quad (5.8)$$

Proof. Lemmas 5.1 and 5.2 give

$$2m_k \leq B = m_k + 3Q_3 + O \leq m_k + 3Q_3 + (1 - m_k - Q_3) = 1 + 2Q_3.$$

Rearranging proves (5.8). $\square$

Combining Theorems 4.2 and 5.3 yields the promised universal estimate.

Corollary 5.4 (triangle-charge inequality). *For every $A \in \Omega_{2k+1}$,*

$$\boxed{m_k(A) - Q_3(A) \leq \frac{1}{2}.} \quad (5.9)$$

6 Proof of the main theorem

We now assemble the short final implication. By (4.3) and (5.9),

$$M + m_k \leq 1 - Q_3 + m_k \leq \frac{3}{2}. \quad (6.1)$$

Lemma 3.2 therefore gives

$$\text{per}(I - A) \leq 2^{k-1}(M + m_k) \leq 2^{k-1} \cdot \frac{3}{2} = 3 \cdot 2^{k-2}. \quad (6.2)$$

It remains to check sharpness. For the matrix $A_\star$ in (1.2), block multiplicativity of the permanent gives

$$\begin{aligned} \text{per}(I - A_\star) &= \text{per}\left(I_3 - \frac{1}{2}(J_3 - I_3)\right) \text{per}(I_2 - P_2)^{k-1} \\ &= \frac{3}{2} 2^{k-1} = 3 \cdot 2^{k-2}. \end{aligned} \quad (6.3)$$

Thus equality is attained, proving the maximum-value part of Theorem 1.1. The argument includes $k = 1$: empty outside matchings have weight one, Q_3 is then the mass of the two oriented 3-cycles, and (6.3) gives $3/2$. $\square$

At $A_\star$, the map masses are

$$(M, m_k, Q_3) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right). \quad (6.4)$$

Consequently both the triangle-charge constant $1/2$ and the compensation constant $3/2$ in (6.1) are sharp.

7 Classification of equality

We now prove the equality assertion in Theorem 1.1. Suppose that $A \in \Omega_{2k+1}$ attains the maximum. The proof above gives the monotone chain

$$\begin{aligned} \text{per}(I - A) &\leq 2^{k-1}(M + m_k) \\ &\leq 2^{k-1} \left(M + Q_3 + \frac{1}{2}\right) \\ &\leq 2^{k-1} \cdot \frac{3}{2}. \end{aligned} \quad (7.1)$$

Equality at the two ends forces equality at every step, and in particular

$$Q_3 = 1 - M. \quad (7.2)$$

The maps counted by M and Q_3 are disjoint subsets of the map space of total mass one in (2.2). Since every map mass is nonnegative, (7.2) has the following support consequence:

Every map f satisfying $a_{i,f(i)} > 0$ for every i either has only even cycles, or is a permutation of cycle type $(3, 2, \dots, 2)$.

We call this property the support dichotomy. Let S be the support digraph of A , with a directed edge $i \rightarrow j$ exactly when $a_{ij} > 0$. Any partial assignment of supported outgoing edges extends to a full positive-mass map: in each unused row, simply choose any positive entry. We use this observation repeatedly.

First, S has no loop. Indeed, a map extending a supported loop has an odd 1-cycle, so it does not have only even cycles, and its fixed point prevents it from having type $(3, 2, \dots, 2)$. The same extension argument shows that S has no directed odd cycle of length at least five.

By the Birkhoff–von Neumann theorem, S contains the edges of a permutation. This permutation has no fixed point. Its cycle lengths sum to the odd number $2k + 1$, so one of them is odd; by the preceding paragraph it must have length three. Thus S contains a directed triangle γ on a vertex triple T .

No supported edge enters T from outside. Otherwise, extend γ together with an edge $z \rightarrow t$, where $z \notin T$ and $t \in T$. The resulting map has the odd cycle γ , but it is not a permutation because t has two distinct preimages. This contradicts the support dichotomy. Each column in T therefore receives its full unit mass from rows in T . Those three column sums exhaust the total mass of the three rows in T , so no supported edge leaves T either. After simultaneous relabelling,

$$A = A_1 \oplus A_2, \quad A_1 \in \Omega_3, \quad A_2 \in \Omega_{2k-2}, \quad (7.3)$$

and both blocks have zero diagonal.

Choose arbitrarily one supported outgoing edge from each row of A_2 and combine these choices with γ . The resulting map contains an odd triangle, so the support dichotomy forces it to be a $(3, 2, \dots, 2)$ -permutation. Consequently every such row choice in the complement is a fixed-point-free involution.

If a complementary row z had two supported targets $t_1 \neq t_2$, choose such an involution g with $g(z) = t_1$. Then $g(t_1) = z$, and $g(t_2) \neq z$. Changing only the choice in row z from t_1 to t_2 would produce another supported row choice, but its edge $z \rightarrow t_2$ would not be reciprocated. This contradicts the preceding universal involution property. Hence every row of A_2 has exactly one positive entry. After relabelling the complementary vertices, row stochasticity and the support dichotomy now give

$$A_2 = P_2^{\oplus(k-1)}. \quad (7.4)$$

Finally, A_1 is a zero-diagonal 3×3 doubly stochastic matrix. Let C be a cyclic permutation matrix. The only fixed-point-free permutations of three vertices are C and $C^\top$, so Birkhoff–von Neumann gives

$$A_1 = xC + (1-x)C^\top, \quad 0 \leq x \leq 1, \quad (7.5)$$

Direct expansion yields

$$\text{per}(I_3 - A_1) = 1 - x^3 - (1-x)^3 + 3x(1-x) = 6x(1-x). \quad (7.6)$$

Using (7.4) and block multiplicativity,

$$\text{per}(I - A) = 6x(1-x)2^{k-1}. \quad (7.7)$$

Equality with $3 \cdot 2^{k-2}$ forces $x(1-x) = 1/4$, hence $x = 1/2$. Thus $A_1 = \frac{1}{2}(J_3 - I_3)$, proving (1.5). The converse is (6.3).

It remains to count the distinct matrices. Choose the three vertices of the half-weight triangle in $\binom{2k+1}{3}$ ways, then choose a perfect matching of the other $2k-2$ vertices in $(2k-3)!!$ ways. The triangle is symmetric, so there is no orientation factor, and distinct choices produce distinct matrices. With the convention $(-1)!! = 1$, this also covers $k = 1$. This completes the proof of Theorem 1.1. $\square$

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