The Marcus–Minc Transform Inequality

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Abstract

We prove the Marcus–Minc conjecture: if $n \geq 2$ is an integer, A is a nonnegative $n \times n$ matrix whose row and column sums equal one, and J_n has every entry $1/n$, then

$$\operatorname{per} A \geq \operatorname{per}\left(\frac{nJ_n - A}{n-1}\right).$$

We also determine all equality cases.

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1 Introduction and main results

For an integer $n \geq 2$, put $[n] = \{1, \dots, n\}$, and let S_n be the set of all permutations of $[n]$. For an $n \times n$ matrix $D = (d_{ij})$, its permanent is

$$\operatorname{per} D = \sum_{\pi \in S_n} \prod_{i=1}^{n} d_{i,\pi(i)}. \tag{1.1}$$

A real matrix is **doubly stochastic** if its entries are nonnegative and the entries in each row and in each column sum to one. We denote the set of $n \times n$ doubly stochastic matrices by Ω_n. Let $\mathbf{1}$ be the all-ones column vector, let

$$J_n = \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathsf{T}},$$

and define the Marcus–Minc transform $\tau_n : \Omega_n \to \Omega_n$ by

$$\tau_n(A) = \frac{nJ_n - A}{n-1}. \tag{1.2}$$

The map is well defined: every entry of a doubly stochastic matrix is at most one, so $nJ_n - A$ is nonnegative, and each of its row and column sums is $n-1$. A **permutation matrix** is a zero–one matrix having exactly one 1 in each row and column.

In 1967 Marcus and Minc [1] conjectured that $\operatorname{per} A \geq \operatorname{per} \tau_n(A)$ for every $A \in \Omega_n$, and that for $n \geq 3$ equality holds only at J_n; we call this the Marcus–Minc conjecture. As reported in [7,9], they proved the inequality for $n = 2$, for symmetric positive semidefinite A, and for A sufficiently close to J_n. Wang proved the inequality in order three and showed that the original equality claim fails [2]: equality also holds for the six matrices $(\mathbf{1}\mathbf{1}^{\mathsf{T}} - P)/2$, with P a permutation matrix. Foregger proved it in order four [3]. Hwang proved it for partly decomposable matrices [4], and Malek proved it near J_n and under a spectral-sector condition [5, 6]. Cheon and Wanless recorded the general assertion as Conjecture 3 in their 2005 update of Minc's catalogue [6]. The problem also appears in later work [7–9].

Our first result proves the Marcus–Minc conjecture and classifies equality in every dimension.

Theorem 1.1. *For every integer $n \geq 2$ and every $A \in \Omega_n$,*

$$\operatorname{per} A \geq \operatorname{per} \tau_n(A). \tag{1.3}$$

The equality cases are exactly the following:

1. *if $n = 2$, every $A \in \Omega_2$;*
2. *if $n = 3$, the matrix J_3 and the six matrices $(\mathbf{1}\mathbf{1}^\mathsf{T} - P)/2$, with P a permutation matrix;*
3. *if $n \geq 4$, only J_n.*

We also give new proofs for the known cases $n = 3, 4$; $n = 2$ is immediate.

The proof comes from two estimates with different dimensional scales. To state them, define the Frobenius norm of $X = (x_{ij})$ by

$$\|X\|_F = \left(\sum_{i,j} x_{ij}^2 \right)^{1/2},$$

and set

$$d_n = \frac{n!}{n^n}, \qquad r^2 = \|A - J_n\|_F^2, \qquad a_n = \frac{1}{2n} + \frac{1}{3n^2}. \tag{1.4}$$

For real t, also put

$$T_n(t) = \sum_{k=0}^{n} \frac{(nt)^k (1-t)^{n-k}}{k!}, \qquad \beta_n = \frac{T_n(1/(n-1)) - 1}{n-1}. \tag{1.5}$$

Theorem 1.2. *For every $n \geq 2$ and every $A \in \Omega_n$,*

$$\frac{\operatorname{per} A}{d_n} \geq e^{a_n r^2}, \qquad \frac{\operatorname{per} \tau_n(A)}{d_n} \leq 1 + \beta_n r^2. \tag{1.6}$$

For $n \geq 5$ one has $\beta_n < 1/(2n) < a_n$. In order four,

$$\frac{1}{8} < \beta_4 = \frac{31}{243} < a_4 = \frac{7}{48}, \qquad a_4 - \beta_4 = \frac{71}{3888}. \tag{1.7}$$

Thus order four genuinely uses the $1/(3n^2)$ term in a_n; the simpler coefficient $1/(2n)$ does not separate the two bounds there.

The bounds also give a global quantitative form of Theorem 1.1. Define

$$G_n(A) = \operatorname{per} A - \operatorname{per} \tau_n(A)$$

and, for $n \geq 5$,

$$\epsilon_n = \frac{1}{2n} - \frac{n}{(n-1)(2(n-2)^2 - n)}, \qquad \epsilon_4 = \frac{71}{3888}. \tag{1.8}$$

We will prove that every ϵ_n in (1.8) is positive. Consequently,

$$G_n(A) \geq d_n \epsilon_n \|A - J_n\|_F^2 \qquad (n \geq 4). \tag{1.9}$$

The constants in (1.9) are explicit rather than asserted to be optimal. The sharper coefficient supplied directly by (1.6) is $a_n - \beta_n$.

Section 2 proves the lower estimate in (1.6). Sections 3 and 4 prove the transformed upper estimate. Section 5 separates the two bounds for every $n \geq 4$. Section 6 treats orders three and two directly and determines all equality cases.

2 The global lower bound

2.1 A coefficient inequality

A polynomial $p \in \mathbb{R}[x_1, \dots, x_m]$ is **real stable** if either p is the zero polynomial or

$$p(z_1, \dots, z_m) \neq 0 \quad \text{whenever} \quad \operatorname{Im} z_1, \dots, \operatorname{Im} z_m > 0.$$

It is **homogeneous of degree** m if every monomial with nonzero coefficient has total degree m. For a homogeneous polynomial of degree m in m variables with nonnegative coefficients, define its capacity by

$$\operatorname{Cap}(p) = \inf_{x_1, \dots, x_m > 0} \frac{p(x_1, \dots, x_m)}{x_1 \cdots x_m}. \tag{2.1}$$

The notation $[x_1 \cdots x_m]p$ means the coefficient of the monomial $x_1 \cdots x_m$ in p. We use the following coefficient bound of Gurvits [10, Theorem 2.4 and Corollary 2.5]. A proof is included because the zero-capacity boundary case requires care and because the bound is the only stable-polynomial input to our argument.

Proposition 2.1 (Gurvits). *If p is real stable, homogeneous of degree m in m variables, and has nonnegative coefficients, then*

$$[x_1 \cdots x_m]p \geq \frac{m!}{m^m} \operatorname{Cap}(p). \tag{2.2}$$

Proof. If $p = 0$ or $m = 1$, the claim is immediate; hence assume $p \neq 0$ and $m \geq 2$. We first recall two closure facts used below. Specializing one variable of a real-stable polynomial at a real value preserves real stability or gives zero: approach that value from the upper half-plane and apply Hurwitz's theorem. Partial differentiation has the same property. Indeed, if $d = \deg_{x_m} p \geq 1$, the leading coefficient $p_d(x_1, \dots, x_{m-1})$ is nonvanishing when all its arguments lie in the upper half-plane: the nonvanishing polynomials $p(z_1, \dots, z_{m-1}, it)/(it)^d$ converge locally uniformly to p_d as $t \to \infty$, and Hurwitz's theorem applies. Thus $p(z_1, \dots, z_{m-1}, \cdot)$ is nonconstant there. Gauss–Lucas places all roots of its derivative in the closed lower half-plane, so the partial derivative is real stable. If $d = 0$, that derivative is zero.

For integers $s \geq 2$, put $g(s) = ((s-1)/s)^{s-1}$, and set $g(1) = 1$.

Let h be a one-variable polynomial of degree at most s, with nonnegative coefficients and all roots real and nonpositive. We first show

$$h'(0) \geq g(s) \inf_{t>0} \frac{h(t)}{t}. \tag{2.3}$$

If $h(0) > 0$, write $h(t) = h(0) \prod_{i=1}^{s} (1 + c_i t)$ with $c_i \geq 0$, adding zero values of c_i when the degree is smaller than s. Set $u = \sum_i c_i = h'(0)/h(0)$. The arithmetic–geometric mean inequality gives

$$h(t) \leq h(0) \left(1 + \frac{ut}{s}\right)^s.$$

If $u > 0$ and $s \geq 2$, evaluate the right-hand side divided by t at $t = s/[u(s-1)]$ to obtain (2.3). If $u = 0$, both sides of (2.3) are zero; if $s = 1$, (2.3) is the direct identity for a linear polynomial. If $h(0) = 0$, nonnegativity of the coefficients gives $\inf_{t>0} h(t)/t = h'(0)$, including the case in which t^2 divides h.

Now fix positive values of $x_1, \ldots, x_{m-1}$ and apply (2.3) to $h(t) = p(x_1, \ldots, x_{m-1}, t)$. Real stability, followed by specialization at positive real values, makes every root of this one-variable polynomial real; nonnegative coefficients make every root nonpositive. If

$$q(x_1, \ldots, x_{m-1}) = \left.\frac{\partial p}{\partial x_m}\right|_{x_m=0},$$

then the definition of capacity and (2.3) give

$$q(x_1, \ldots, x_{m-1}) \geq g(m) \operatorname{Cap}(p) x_1 \cdots x_{m-1}.$$

Hence $\operatorname{Cap}(q) \geq g(m) \operatorname{Cap}(p)$. The closure facts above allow the same step to be iterated. At each stage the symbol s in $g(s)$ is the number of variables still present. After the last differentiation the resulting constant is $[x_1 \cdots x_m]p$, and

$$\prod_{s=2}^{m} g(s) = \frac{m!}{m^m}.$$

This proves (2.2), including zero capacity. □

2.2 A one-column bound

Fix $A = (a_{ij}) \in \Omega_n$ whose entries are initially all positive, and fix a column index j. Write

$$c_i = a_{ij}, \qquad L_i(x) = \sum_{k \neq j} a_{ik} x_k.$$

Consider

$$p_A(x) = \prod_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} x_k \right).$$

Every linear factor is nonzero when all variables have positive imaginary parts, so p_A is real stable. Differentiation and specialization give the nonzero real-stable polynomial

$$q_j(x) = \left.\frac{\partial p_A}{\partial x_j}\right|_{x_j=0} = \sum_{i=1}^{n} c_i \prod_{s \neq i} L_s(x). \tag{2.4}$$

It is homogeneous of degree $n-1$ in the $n-1$ variables indexed by $k \neq j$, and the coefficient of $\prod_{k \neq j} x_k$ is per A.

Since $\sum_i c_i = 1$, weighted arithmetic–geometric mean applied to (2.4) gives

$$q_j(x) \geq \prod_{i=1}^{n} \left(\prod_{s \neq i} L_s(x) \right)^{c_i} = \prod_{s=1}^{n} L_s(x)^{1-c_s}. \tag{2.5}$$

For each row s, the remaining row sum is $\sum_{k \neq j} a_{sk} = 1 - c_s$. A second weighted arithmetic–geometric mean inequality gives

$$L_s(x)^{1-c_s} \geq (1-c_s)^{1-c_s} \prod_{k \neq j} x_k^{a_{sk}}. \tag{2.6}$$

Multiplying (2.6) over s and using the remaining column sums shows

$$\operatorname{Cap}(q_j) \geq \prod_{i=1}^{n} (1-c_i)^{1-c_i}.$$

Proposition 2.1, with $m = n - 1$, therefore yields

$$\operatorname{per} A \geq d_{n-1} \prod_{i=1}^{n} (1 - a_{ij})^{1-a_{ij}}. \tag{2.7}$$

The full one-column bound (2.7) is implicit in the proof of Laurent and Schrijver [11, proof of Corollary 1d, equation (39)]; our use of it is the explicit quadratic estimate (2.11). For an arbitrary $A \in \Omega_n$, apply (2.7) to $(1-\delta)A + \delta J_n$ and let $\delta \downarrow 0$. We use the continuous convention $0^0 = 1$. This proves (2.7) on the boundary without assuming continuity or attainment of capacity.

2.3 From one column to the global lower bound

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = (1 - x) \log(1 - x),$$

with its continuous value $f(1) = 0$, and define the squared deviation of column j from the uniform column by

$$F_j = \sum_{i=1}^{n} \left(a_{ij} - \frac{1}{n}\right)^2.$$

Because $d_{n-1}(1 - 1/n)^{n-1} = d_n$, taking logarithms in (2.7) gives

$$\log \frac{\operatorname{per} A}{d_n} \geq \sum_{i=1}^{n} f(a_{ij}) - n f(1/n). \tag{2.8}$$

The convergent endpoint-valid expansion

$$f(x) = -x + \sum_{k \geq 2} \frac{x^k}{k(k-1)} \tag{2.9}$$

makes the gain over the uniform column explicit. Put $s_2 = \sum_i a_{ij}^2 = 1/n + F_j$. Jensen's inequality with weights a_{ij} gives

$$\sum_i a_{ij}^3 \geq s_2^2, \qquad \sum_i a_{ij}^3 - \frac{1}{n^2} \geq \frac{2F_j}{n} + F_j^2. \tag{2.10}$$

For every $k \geq 4$, ordinary convexity gives $\sum_i a_{ij}^k \geq n^{1-k}$. Substituting (2.9) into the difference in (2.8), the linear terms cancel, the quadratic term contributes $F_j/2$, and (2.10) controls the cubic term. Consequently

$$\sum_i f(a_{ij}) - n f(1/n) \geq \left(\frac{1}{2} + \frac{1}{3n}\right) F_j + \frac{F_j^2}{6} \geq \left(\frac{1}{2} + \frac{1}{3n}\right) F_j. \tag{2.11}$$

Since $\sum_j F_j = r^2$, some column has $F_j \geq r^2/n$. Equations (2.8)–(2.11) prove the first inequality in (1.6):

$$\frac{\operatorname{per} A}{d_n} \geq \exp\left[\left(\frac{1}{2n} + \frac{1}{3n^2}\right) r^2\right]. \tag{2.12}$$

3 A subpermanent estimate for zero-margin matrices

For subsets $S, T \subseteq [n]$ of the same size, let $X[S,T]$ be the submatrix of X with row set S and column set T. For $0 \le k \le n$, define the sum of all $k \times k$ subpermanents by

$$\sigma_k(X) = \sum_{\substack{S,T\subseteq[n]\\|S|=|T|=k}} \operatorname{per} X[S,T], \qquad \sigma_0(X) = 1. \tag{3.1}$$

A matrix has **zero margins** if every row sum and every column sum is zero. Its Euclidean operator norm is

$$\|X\|_{2\to2} = \sup_{v\neq0} \frac{\|Xv\|_2}{\|v\|_2}.$$

Write I_n for the identity matrix and set

$$Q = I_n - J_n.$$

Put $H = \mathbf{1}^\perp = \{v \in \mathbb{R}^n : \mathbf{1}^\mathsf{T} v = 0\}$. The map Q is the identity on H and is zero on the line $\mathbb{R}\mathbf{1}$; this is what we mean by the orthogonal projection onto H.

Lemma 3.1. *If X is a real $n \times n$ matrix with zero margins and $\|X\|_{2\to2} \le 1$, then, for $2 \le k \le n$,*

$$|\sigma_k(X)| \le \frac{\sigma_k(Q)}{n-1}\|X\|_F^2. \tag{3.2}$$

Proof. We use tensor powers of $\mathbb{R}^n$ with the product inner product

$$\langle x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k\rangle = \prod_{\ell=1}^{k} \langle x_\ell, y_\ell\rangle,$$

extended bilinearly. If $S = \{i_1, \ldots, i_k\}$ is a k-element subset of $[n]$, put

$$v_S = \frac{1}{\sqrt{k!}} \sum_{\pi\in S_k} e_{i_{\pi(1)}} \otimes \cdots \otimes e_{i_{\pi(k)}}, \qquad u_k = \sum_{|S|=k} v_S, \tag{3.3}$$

where $e_1, \ldots, e_n$ are the standard basis vectors. The vectors v_S are orthonormal, and expanding the inner product gives

$$\langle v_S, X^{\otimes k} v_T\rangle = \operatorname{per} X[S,T]. \tag{3.4}$$

Zero margins imply $X = QXQ$. Hence, with $w = Q^{\otimes k} u_k \in H^{\otimes k}$,

$$\sigma_k(X) = \langle w, X^{\otimes k} w\rangle, \qquad \|w\|^2 = \sigma_k(Q) \ge 0. \tag{3.5}$$

The second identity follows because $Q^{\otimes k}$ is an orthogonal projection. It also supplies the nonnegativity of every $\sigma_k(Q)$ used below.

We now define the tensor property responsible for the estimate. A linear operator R on H is **self-adjoint** if $\langle Rx, y\rangle = \langle x, Ry\rangle$ for all $x, y \in H$. For $w \in H^{\otimes k}$ and $1 \le \ell \le k$, its ℓ-th **one-factor marginal** is the unique self-adjoint operator ρ_ℓ on H satisfying

$$\operatorname{tr}(\rho_\ell R) = \left\langle w, (I_H^{\otimes(\ell-1)} \otimes R \otimes I_H^{\otimes(k-\ell)})w\right\rangle \tag{3.6}$$

for every self-adjoint operator R on H; here tr denotes the sum of the diagonal entries of an operator in an orthonormal basis. We call w **one-factor isotropic** when each ρ_ℓ is a scalar multiple of I_H.

To see existence, choose an orthonormal basis (h_a) of H and, after putting the ℓ-th factor first, write $w = \sum_a h_a \otimes z_a$, where $z_a \in H^{\otimes(k-1)}$. The matrix $(\langle z_a, z_b\rangle)_{a,b}$ is self-adjoint and satisfies (3.6), so it defines ρ_ℓ. If two self-adjoint operators satisfy (3.6), their difference D gives $\operatorname{tr}(D^2) = 0$ upon taking $R = D$, and therefore $D = 0$.

Simultaneously permuting the n coordinate labels fixes u_k, commutes with Q, and therefore fixes w. Thus each ρ_ℓ commutes with every coordinate permutation on H. Extend ρ_ℓ by zero on $\mathbb{R}\mathbf{1}$. A matrix commuting with every permutation has one constant on its diagonal and another off its diagonal. Its restriction to H is therefore scalar. Since $\operatorname{tr}\rho_\ell = \|w\|^2$, we obtain

$$\rho_\ell = \frac{\|w\|^2}{n-1} I_H \qquad (1 \le \ell \le k). \tag{3.7}$$

This is the asserted isotropy; it is a property of w, not an assumption on X.

Choose linear operators U, V on H satisfying $U^\mathsf{T} U = V^\mathsf{T} V = I_H$ and numbers $0 \le s_1, \dots, s_{n-1} \le 1$ such that

$$X|_H = UDV^\mathsf{T}, \qquad D = \operatorname{diag}(s_1, \dots, s_{n-1}).$$

Such a factorization is a singular-value decomposition; the displayed identities say that U and V are orthogonal, and the upper bounds on the s_i follow from $\|X\|_{2\to 2} \le 1$. Put

$$a = (U^\mathsf{T})^{\otimes k} w, \qquad b = (V^\mathsf{T})^{\otimes k} w, \qquad K = D^{\otimes k}.$$

The one-factor marginals of both a and b still equal (3.7). A self-adjoint operator R is **positive semidefinite** if $\langle z, Rz\rangle \ge 0$ for every z. For self-adjoint operators R, S, write $R \preceq S$ when $S - R$ is positive semidefinite. For every diagonal entry of K, the assumptions $k \ge 2$ and $0 \le s_i \le 1$ give

$$\prod_{\ell=1}^{k} s_{i_\ell} \le \left(\prod_{\ell=1}^{k} s_{i_\ell}^2\right)^{1/k} \le \frac{1}{k}\sum_{\ell=1}^{k} s_{i_\ell}^2.$$

Equivalently,

$$0 \preceq K \preceq \frac{1}{k}\sum_{\ell=1}^{k} I_H^{\otimes(\ell-1)} \otimes D^2 \otimes I_H^{\otimes(k-\ell)}. \tag{3.8}$$

Equations (3.6)–(3.8) yield

$$\langle a, Ka\rangle,\ \langle b, Kb\rangle \le \frac{\|w\|^2}{n-1}\operatorname{tr} D^2.$$

Cauchy–Schwarz for the positive semidefinite operator K now gives

$$\begin{aligned} |\sigma_k(X)| &= |\langle a, Kb\rangle| \\ &\le \sqrt{\langle a, Ka\rangle\langle b, Kb\rangle} \\ &\le \frac{\sigma_k(Q)}{n-1}\|X\|_F^2, \end{aligned}$$

because $\operatorname{tr} D^2 = \|X\|_F^2$. This proves (3.2). □

For later use, we verify the contraction hypothesis. If $A \in \Omega_n$ and $x \in \mathbb{R}^n$, Jensen's inequality in each row and the column sums give

$$\sum_i \left(\sum_j a_{ij} x_j\right)^2 \le \sum_{i,j} a_{ij} x_j^2 = \sum_j x_j^2. \tag{3.9}$$

Thus $\|A\|_{2\to 2} \le 1$. With $B = A - J_n$, the row and column sum identities also give $B = QAQ$, so B has zero margins and $\|B\|_{2\to 2} \le 1$. Lemma 3.1 applies to B.

4 The transformed upper bound

Let $B = A - J_n$. Expanding the permanent according to the entries chosen from tB gives, for every real t,

$$\operatorname{per}(J_n + tB) = \sum_{k=0}^{n} \frac{(n-k)!}{n^{n-k}} t^k \sigma_k(B). \tag{4.1}$$

Indeed, after choosing the k rows and k columns used by B, the remaining $n-k$ rows can be matched to the remaining columns in $(n-k)!$ ways, and each remaining entry of J_n contributes $1/n$. Since B has zero margins, $\sigma_1(B) = 0$.

Put $q = 1/(n-1)$. Formula (1.2) becomes $\tau_n(A) = J_n - qB$. Applying Lemma 3.1 to every term of (4.1), and bounding the odd terms in absolute value rather than assuming a sign, gives

$$\begin{aligned} \operatorname{per} \tau_n(A) &\le d_n + \frac{r^2}{n-1} \sum_{k=2}^{n} \frac{(n-k)!}{n^{n-k}} q^k \sigma_k(Q) \\ &= d_n + \frac{r^2}{n-1} (\operatorname{per}(J_n + qQ) - d_n). \end{aligned} \tag{4.2}$$

Here $\sigma_1(Q) = 0$ because Q has zero margins, and $\sigma_k(Q) \ge 0$ by (3.5). No positivity of an odd-degree coefficient $\sigma_k(B)$ is being assumed.

To evaluate the last permanent, write $J_n + tQ = tI_n + (1-t)J_n$. Choosing the k rows in which the identity term is used gives

$$\operatorname{per}(J_n + tQ) = \sum_{k=0}^{n} \binom{n}{k} t^k (n-k)! \left(\frac{1-t}{n}\right)^{n-k}. \tag{4.3}$$

Dividing (4.3) by d_n gives exactly $T_n(t)$ from (1.5). Therefore (4.2) is the second inequality in (1.6):

$$\frac{\operatorname{per} \tau_n(A)}{d_n} \le 1 + \beta_n r^2. \tag{4.4}$$

5 Separation in orders at least four

Assume first that $n \ge 5$, and retain $q = 1/(n-1)$. Formula (1.5) can be written as the truncated-exponential identity

$$T_n(q) = (1-q)^n \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{nq}{1-q}\right)^k. \tag{5.1}$$

All omitted terms of the exponential series are positive. Hence

$$T_n(q) < \exp\left(n \left[\log(1-q) + \frac{q}{1-q}\right]\right). \tag{5.2}$$

The expression in brackets satisfies

$$\log(1-q) + \frac{q}{1-q} = \int_0^q \frac{t}{(1-t)^2}\, dt \le \frac{q^2}{2(1-q)^2}. \tag{5.3}$$

Set $x = n/[2(n-2)^2]$. Then $0 < x \le 5/18 < 1$, and (5.2)–(5.3), together with $e^x < 1/(1-x)$, imply

$$T_n(q) < e^x < \frac{1}{1-x}, \qquad \beta_n < \frac{n}{(n-1)(2(n-2)^2 - n)}. \tag{5.4}$$

The rational expression on the right of (5.4) is smaller than $1/(2n)$. After multiplication by the positive denominators, this is equivalent to

$$\begin{aligned} 2n^3 - 13n^2 + 17n - 8 = 2(n-5)^3 + 17(n-5)^2 \\ + 37(n-5) + 2 > 0. \end{aligned} \tag{5.5}$$

This proves $\beta_n < 1/(2n)$ and the positivity of ϵ_n in (1.8) for every $n \geq 5$. If $A \neq J_n$, then $r > 0$, so (2.12), (4.4), and (5.4) give the strict chain

$$\frac{\operatorname{per} A}{d_n} \geq e^{r^2/(2n)} > 1 + \frac{r^2}{2n} > 1 + \beta_n r^2 \geq \frac{\operatorname{per} \tau_n(A)}{d_n}. \tag{5.6}$$

At $A = J_n$, the two permanents both equal d_n.

For $n = 4$, direct evaluation of (1.5) gives

$$T_4(t) = 1 + 2t^2 + \frac{8}{3}t^3 + 5t^4, \qquad T_4(1/3) = \frac{112}{81}.$$

Consequently

$$\beta_4 = \frac{31}{243}, \qquad a_4 = \frac{7}{48}, \qquad a_4 - \beta_4 = \frac{71}{3888} > 0,$$

which proves (1.7) and makes the comparison in (1.3) strict away from J_4.

Finally, $e^{a_n r^2} \geq 1 + a_n r^2$. Subtracting the two bounds in (1.6) therefore gives

$$G_n(A) \geq d_n(a_n - \beta_n)r^2. \tag{5.7}$$

For $n = 4$, (5.7) is (1.9). For $n \geq 5$, replacing a_n by the smaller $1/(2n)$ and using (5.4) gives (1.9) with the explicit ϵ_n from (1.8). This establishes (1.3) and its equality case for every $n \geq 4$.

6 Orders three and two

6.1 Order three

Let $A \in \Omega_3$, put $B = A - J_3$, and retain $r^2 = \|B\|_F^2$. Direct expansion of a 3×3 zero-margin matrix gives

$$\sigma_2(B) = \frac{1}{2}r^2, \qquad \operatorname{per} B = \frac{2}{3}\sum_{i,j=1}^{3} b_{ij}^3. \tag{6.1}$$

Apply (4.1) at $t = 1$ and $t = -1/2$. Since $\tau_3(A) = J_3 - B/2$, one obtains

$$G_3(A) = \frac{1}{8}\left(r^2 + 6\sum_{i,j=1}^{3} b_{ij}^3\right). \tag{6.2}$$

Consider one row (x, y, z) of A. It consists of nonnegative numbers with $x + y + z = 1$. Its contribution to the expression in parentheses in (6.2) is

$$\begin{aligned} \sum_{u\in\{x,y,z\}} (u - 1/3)^2 + 6 \sum_{u\in\{x,y,z\}} (u - 1/3)^3 \\ = 2(1 - 4(xy + xz + yz) + 9xyz). \end{aligned} \tag{6.3}$$

Reorder the three entries so that $x \geq y \geq z$. The following elementary form of Schur's identity, using $x+y+z=1$, is

$$1 - 4(xy + xz + yz) + 9xyz = (x-y)^2(x+y-z) + z(x-z)(y-z). \tag{6.4}$$

Both terms on the right are nonnegative: in particular, $x+y-z = 1-2z \geq 1/3$. Thus every row contribution in (6.3) is nonnegative, which proves $G_3(A) \geq 0$.

Equation (6.4) also determines equality. It first forces $x = y$ and then forces either $z = 0$ or $z = x$. Thus each equality row is either $(1/3, 1/3, 1/3)$ or a permutation of $(1/2, 1/2, 0)$. If exactly one or two rows are half-rows, the half-rows would have to contribute respectively $1/3$ or $2/3$ to every column after the uniform rows are removed. Their contribution to each column is a multiple of $1/2$, so this is impossible. All rows are therefore uniform, giving J_3, or all rows are half-rows. In the latter case every column contains exactly one zero, the zero positions form a permutation matrix P, and

$$A = \frac{\mathbf{1}\mathbf{1}^{\mathsf{T}} - P}{2}.$$

Conversely, J_3 and each of these six matrices make every row contribution in (6.3) zero. This proves the order-three statement and its full equality classification.

6.2 Order two

Every matrix in Ω_2 has the form

$$A = \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix}, \qquad 0 \leq t \leq 1.$$

The transform replaces t by $1-t$, while

$$\operatorname{per} A = t^2 + (1-t)^2.$$

Hence every $A \in \Omega_2$ is an equality case.

6.3 Completion of the proof

Proof of Theorem 1.1. Section 5 proves (1.3) for $n \geq 4$, with equality only at J_n. Section 6.1 proves (1.3) in order three, with exactly the seven equality cases listed in the theorem. Section 6.2 proves that every matrix in Ω_2 is an equality case. □

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