A linear gap for the maximal cp-rank

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Abstract

We prove that the maximal cp-rank p_n satisfies $p_n \geq n(n-3)/2$ for odd $n \geq 5$ and $p_n \geq n(n-3)/2-1$ for even $n \geq 6$. Together with the known upper bound $p_n \leq \binom{n+1}{2} - 4$, this gives $p_n = n^2/2 + O(n)$.

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1 Introduction

A real symmetric $n \times n$ matrix A is **completely positive** if $A = BB^{\mathsf{T}}$ for an entrywise nonnegative real matrix B with finitely many columns. Its **cp-rank** $\operatorname{cpr}(A)$ is the least possible number of columns of a matrix B in such a representation; set $\operatorname{cpr}(0) = 0$. Write CP_n for the cone of these matrices and

$$p_n = \max\{\operatorname{cpr}(A) : A \in \mathrm{CP}_n\}. \tag{1.1}$$

Write $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for every } i\}$; thus $x \geq 0$ means that every coordinate of x is nonnegative. A real symmetric $n \times n$ matrix S is **copositive** if $x^{\mathsf{T}} S x \geq 0$ for every $x \in \mathbb{R}^n_+$.

Drew, Johnson, and Loewy conjectured $p_n = \lfloor n^2/4 \rfloor$ for $n \geq 4$ [1]. Bomze, Schachinger, and Ullrich disproved this at orders 7–11 [5] and at every order $n \geq 12$ [3]. The general upper bound $p_n \leq \binom{n+1}{2} - 4$ for $n \geq 5$ is [2, Corollary 5.2]. Relative to this upper bound, the lower bounds of [3] and [4, Theorem 7.4] leave gaps of $O(n^{3/2})$ and $O(n \log\log n)$, respectively.

We follow [5, Sections 2–3] and [3, Lemmas 2.1–2.2 and Example 3.1] in using copositive matrices with nonzero vectors $u \geq 0$ satisfying $u^{\mathsf{T}} S u = 0$ and in deleting the same coordinates from rows and columns. Let $\widehat{S}$ be the order-seven matrix of [5, Example 1]. With zero-based indices modulo 7, the choice $t = 4\pi - 7\arccos(-1/6)$ in our family gives $S_{ij} = \widehat{S}_{2i,2j}/162$. Here we give a uniform family in every odd order, prove its copositivity, classify all such vectors up to positive scaling, and obtain explicit lower bounds in both parities without finite-order computation.

Theorem 1.1. *For every integer $n \geq 5$, put $\varepsilon_n = 0$ if n is odd and $\varepsilon_n = 1$ if n is even. Then*

$$\boxed{\frac{n(n-3)}{2} - \varepsilon_n \;\leq\; p_n \;\leq\; \binom{n+1}{2} - 4.} \tag{1.2}$$

Thus $p_n = \frac{1}{2}n^2 + O(n)$. The difference between the upper and lower expressions in (1.2) is $2n - 4 + \varepsilon_n$. The construction gives the lower expression on a particular subset of CP_n; (1.2) does not assert an exact value of p_n.

At $n = 5$, our bound 5 compares with $p_5 = 6$ [2, Remark 5.1]. At $n = 6$, our bound 8 compares with $9 \le p_6 \le 15$ [3, Table 2] and [2, Theorem 6.1]. At $n = 7, 13$ our bounds match published lower bounds $14, 65$ [5, Example 1] and [3, Table 1]; at $n = 14, 15$ they give $76, 90$, below $80, 95$ [3, Table 1]. For $n = 8, \ldots, 12$, our values $19, 27, 34, 44, 53$ exceed the published $18, 26, 30, 37, 50$ [5, Examples 2–3], [3, Table 1], and [4, Table 1].

The proof classifies the finitely many nonnegative vectors $u \neq 0$ with $u^\mathsf{T} S u = 0$ for a suitable copositive S, up to positive scaling. Every nonzero factor column of a completely positive matrix orthogonal to S is a positive multiple of one of them. Lemma 2.1 shows that the dimension of the span of their outer products is the maximum cp-rank among these matrices; Sections 3–5 compute it in odd and even order.

2 Finite zero sets and cp-rank

Let S be any real symmetric $n \times n$ matrix. A nonzero vector $u \in \mathbb{R}^n_+$ with $u^\mathsf{T} S u = 0$ is a **nonnegative zero** of S. Its **support** is $\operatorname{supp}(u) = \{i : u_i > 0\}$. The ray generated by a nonnegative zero u is $\{cu : c \ge 0\}$; two nonnegative zeros determine the same ray exactly when one is a positive multiple of the other. We call such a ray a **zero ray** of S.

On the space of real symmetric $n \times n$ matrices, we use the **Frobenius inner product** $\langle S, A \rangle = \operatorname{tr}(SA)$. An **exposed face** of a cone K is its intersection with the zero set of a linear functional that is nonnegative on K. If S is copositive, then for every $A = BB^\mathsf{T} \in \mathrm{CP}_n$, writing the columns of B as $b_1, \ldots, b_s$ gives $\langle S, A \rangle = \sum_{j=1}^s b_j^\mathsf{T} S b_j \ge 0$. Hence the following set is an exposed face of the completely positive cone CP_n:

$$F(S) = \{A \in \mathrm{CP}_n : \langle S, A \rangle = 0\}. \tag{2.1}$$

The following finite-zero principle appears in [5, Lemma 2.1] and [3, Lemmas 2.1–2.2].

Lemma 2.1 (finite-zero face)**.** *Suppose a copositive S has exactly $q < \infty$ nonnegative zero rays, represented by $u^{(1)}, \ldots, u^{(q)}$. If*

$$h = \dim \operatorname{span}\{u^{(a)} u^{(a)\mathsf{T}} : 1 \le a \le q\}, \tag{2.2}$$

then the maximum cp-rank of matrices in $F(S)$ is h.

Proof. If $q = 0$, then $F(S) = \{0\}$ and $h = 0$. Otherwise, choose a factorization $A = BB^\mathsf{T}$ for $A \in F(S)$. Copositivity makes every summand's pairing with S nonnegative, so each nonzero column of B lies on a listed zero ray. Thus $F(S)$ is generated by their outer products. The conic form of Carathéodory's theorem says that any point of a cone spanning an h-dimensional space is a nonnegative combination of at most h generators, so cp-rank is at most h. A point of $F(S)$ outside the finitely many spans of fewer than h generators has cp-rank h. Such a point exists because a finite union of proper linear subspaces cannot cover an h-dimensional cone.

□

3 A copositive family in odd order

Let $n = 2m + 1 \ge 5$ and index coordinates by $\mathbb{Z}/n\mathbb{Z}$ in cyclic order. We also regard index i as the vertex at polar angle $2\pi i/n$ of the unit regular n-gon. The **cyclic distance** $d(i, j)$ is the smaller number of steps along the polygon from i to j, so $1 \le d(i, j) \le m$ for $i \neq j$. A **cyclic interval of L steps** is $\{i, i + 1, \ldots, i + L\}$ modulo n, where $0 \le L < n$. Fix

$$0 < t < \frac{2\pi}{2n-3}, \qquad a = \frac{2\pi + 3t}{n}, \qquad \phi_k = ak - t \quad (1 \le k \le m), \tag{3.1}$$

and define a real symmetric matrix S by

$$S_{ii} = 1, \qquad S_{ij} = \cos \phi_{d(i,j)} \quad (i \ne j). \tag{3.2}$$

At $n = 5$, the matrices in (3.2) belong, up to a permutation of coordinates, to Hildebrand's order-five copositive family [6].

This matrix is **circulant**: its entries satisfy $S_{i+h,j+h} = S_{ij}$ for every $h \in \mathbb{Z}/n\mathbb{Z}$.

The interval in (3.1) gives $0 < \phi_1 < \cdots < \phi_m < \pi$. Indeed $\phi_1 = (2\pi - (n-3)t)/n > 0$ and $\phi_m = \pi - \pi/n + (n-3)t/(2n) < \pi$. The **pair angle** of $\{i, j\}$ is $\phi_{d(i,j)}$. For an index set I, the **principal submatrix** on I is $S_I = (S_{ij})_{i,j \in I}$.

For three distinct vertices v_1, v_2, v_3 in cyclic order, choose representatives $0 \le v_1 < v_2 < v_3 < n$. Their **cyclic gaps** are $k = v_2 - v_1$, $\ell = v_3 - v_2$, and $r = n + v_1 - v_3$: the positive numbers of steps from v_1 to v_2, from v_2 to v_3, and from v_3 back to v_1. They satisfy $k + \ell + r = n$; starting the cyclic listing at a different vertex only permutes them cyclically. We call the triple **surrounding** if $k, \ell, r \le m$. Then each gap is the cyclic distance of its endpoint pair, and the three pair angles sum to $a(k + \ell + r) - 3t = an - 3t = 2\pi$.

For example, let $n = 7$, $m = 3$, $t = \pi/9$ (which satisfies (3.1)), and therefore $a = \pi/3$. The triple $\{0, 2, 4\}$ has gaps $(k, \ell, r) = (2, 2, 3)$, so it is surrounding. Its pair distances are $d(0, 2) = d(2, 4) = 2$ and $d(4, 0) = 3$; the pairs $\{0, 2\}$ and $\{2, 4\}$ each have angle $\phi_2 = 5\pi/9$, while $\{0, 4\}$ has angle $\phi_3 = 8\pi/9$. These three angles sum to 2π.

Lemma 3.1 (triangle zeros)**.** *Let $T = \{v_1, v_2, v_3\}$ be a surrounding triple, with its vertices listed in cyclic order and gaps k, ℓ, r as above. Up to multiplication by a positive scalar, there is exactly one nonnegative zero u of S with* $\operatorname{supp}(u) = T$*. A representative, with all other coordinates zero, is*

$$u_{v_1} = \sin \phi_\ell, \qquad u_{v_2} = \sin \phi_r, \qquad u_{v_3} = \sin \phi_k, \tag{3.3}$$

Proof. For the vertices in cyclic order, take planar unit vectors w_1, w_2, w_3 at angles $0, \phi_k, \phi_k + \phi_\ell$. Their **Gram matrix**, the matrix $(w_a \cdot w_b)_{a,b=1}^3$ of pairwise inner products, is S_T, and $\sin \phi_\ell w_1 + \sin \phi_r w_2 + \sin \phi_k w_3 = 0$: the vertical coordinate is $\sin \phi_k (\sin \phi_r + \sin(\phi_k + \phi_\ell)) = 0$, and the horizontal coordinate is $\sin \phi_\ell + \sin(\phi_k + \phi_r) = 0$, using $\phi_k + \phi_\ell + \phi_r = 2\pi$. Thus (3.3) is the unique nonnegative kernel ray of the rank-two positive semidefinite matrix S_T. □

In the seven-vertex example, (3.3) gives $u_0 = u_4 = \sin(5\pi/9)$ and $u_2 = \sin(8\pi/9)$, with all other coordinates zero.

We now prove that S is copositive and has no other zeros. An **open semicircle** means an open arc of angular length π on the unit circle. The next lemma concerns the regular n-gon defined above.

Lemma 3.2 (four-vertex geometry)**.** *A vertex set containing no surrounding triple lies in an open semicircle, hence in a cyclic interval of at most m steps. A four-vertex set containing a surrounding triple contains exactly two such triples. They share a pair $\{i, j\}$.*

Proof. A surrounding triple is precisely a triple of polygon vertices whose convex hull contains the origin: three points on a circle enclose its center exactly when they do not fit in an open semicircle, equivalently when none of their cyclic gaps exceeds m. If the origin were in the convex hull of a vertex set, planar Carathéodory's theorem would put it in the convex hull of at most three of those vertices. One vertex cannot contain the origin, and a segment between two vertices could contain it only if they were antipodal, impossible for odd n. Hence a set with no surrounding triple has a convex hull omitting the origin. A linear functional strictly positive on this convex hull places all its vertices in an open half-plane through the origin, hence in an open semicircle spanning at most m cyclic steps.

For four vertices let g_1, g_2, g_3, g_4 be their cyclic gaps, summing to $2m+1$. If any $g_i > m$, no triple is surrounding. Otherwise the triple obtained by omitting a vertex is surrounding exactly when the two gaps merged by that omission sum to at most m. The sums g_1+g_2 and g_3+g_4 add to $2m+1$, so exactly one is at most m; the same holds for g_2+g_3 and g_4+g_1. Hence exactly two triples are surrounding. They share a pair. $\square$

Lemma 3.3 (overlapping triangle zeros)**.** *If u, v represent the rays of the two surrounding triples in a four-set, then $u^\mathsf{T} S v > 0$.*

Proof. Because S depends only on cyclic distance, rotations and reflections preserve it. Relabel the common pair as $\{0, k\}$ with $1 \le k \le m$. A third vertex of a surrounding triple through this pair lies in $\{m+1, \ldots, m+k\}$. Write the third vertices of u, v as p, q, respectively, and set $d = d(p,q) = |p-q|$, so $1 \le d \le m-1$. Since $1 \le p-k, n-p \le m$, we have $ap - 2t = \phi_k + \phi_{p-k} = 2\pi - \phi_{n-p}$, and likewise for q. Choose planar unit vectors at angles $0, \phi_k, ap-2t, aq-2t$, indexed by $0, k, p, q$, and let G be their Gram matrix. Since $an = 2\pi + 3t$, its entries agree with S on each surrounding triple, while $G_{pq} = \cos(ad)$. Each of u, v is a dependence among its triple's planar vectors, so $u^\mathsf{T} G v = 0$. The only entry at which S and G can differ in $u^\mathsf{T} S v$ is the pair $\{p, q\}$:

$$u^\mathsf{T} S v = u_p v_q[\cos(ad-t) - \cos(ad)] = 2u_p v_q \sin(ad - t/2)\sin(t/2) > 0. \tag{3.4}$$

Here $ad - t/2 > 0$ follows from $a > t$, and $ad < \pi$ follows from $d \le m-1$ and $t < 2\pi/(n-3)$, a weaker consequence of (3.1). $\square$

Proposition 3.4 (complete zero list)**.** *The matrix S is copositive, and its nonnegative zero rays are exactly those in Lemma 3.1.*

Proof. Minimize $x^\mathsf{T} S x$ on the simplex $\Delta = \{x \ge 0 : \sum_i x_i = 1\}$, and let I be the support of a minimizer. If I contains no surrounding triple, Lemma 3.2 places it in a cyclic interval $\{s, s+1, \ldots, s+L\}$ modulo n, with $L \le m$. For each $i \in I$, choose the unique $r_i \in \{0, \ldots, L\}$ with $i \equiv s + r_i \pmod{n}$, and let $G_{ij} = \cos(a(r_i - r_j))$, a planar Gram matrix. For $i \ne j$ in the interval, $d = |r_i - r_j| \le m$ and

$$S_{ij} - G_{ij} = 2\sin(ad - t/2)\sin(t/2) > 0. \tag{3.5}$$

Indeed $a > t$ and $am - t/2 = \pi - \pi/n + (2n-3)t/(2n) < \pi$. The diagonals agree. Since G is positive semidefinite, the off-diagonal correction gives $x^\mathsf{T} S x > 0$ for supports of size at least two; singleton supports have value one.

If I is a surrounding triple, S_I is positive semidefinite by Lemma 3.1. If I contains a surrounding triple and at least one more vertex, choose a four-set inside I containing it. Let u, v be its two triangle rays, each normalized to have coordinate sum one. Then $y = u - v$ has coordinate sum zero, and Lemma 3.3 gives $y^\mathsf{T} S y = -2u^\mathsf{T} S v < 0$. Both $x + \epsilon y$ and $x - \epsilon y$ remain in Δ for small $\epsilon > 0$, but the average of their quadratic values is $x^\mathsf{T} S x + \epsilon^2 y^\mathsf{T} S y < x^\mathsf{T} S x$, contradicting minimality. Hence the minimum is nonnegative and S is copositive.

A normalized zero minimizes on Δ, so the cases force its support to be a surrounding triple; S_I then forces (3.3). $\square$

4 The odd-order face dimension

Index the rows of M and H by surrounding triples T. Index the columns of the zero–one matrix M by unordered pairs $e = \{i, j\}$ with $0 \le i < j < n$, and set $M_{T,e} = 1$ if $e \subset T$ and $M_{T,e} = 0$ otherwise. For each T, let $u^{(T)}$ be the representative in (3.3), with $u_i^{(T)} = 0$ for $i \notin T$. Index the columns of H by (i, j) with $0 \le i \le j < n$ and define

$$H_{T,(i,j)} = u_i^{(T)} u_j^{(T)} \qquad (0 \le i \le j < n).$$

Thus (i, i) labels a diagonal coordinate, while for $i < j$ the column (i, j) of H corresponds to the column $\{i, j\}$ of M. The rank of H is the dimension h in (2.2).

Lemma 4.1 (rank transfer)**.** $\operatorname{rank} H = \operatorname{rank} M$.

Proof. For a triple $T = \{i, j, k\}$, let $g_T = \sin\phi_{d(i,j)} \sin\phi_{d(i,k)} \sin\phi_{d(j,k)} > 0$. For a pair $e = \{a, b\}$ write $d(e) = d(a, b)$, and let H_{off} be the block of H with only off-diagonal columns. If $e \subset T$, equation (3.3) gives $u_a u_b \sin\phi_{d(e)} = g_T$. Thus

$$H_{\text{off}} = \operatorname{diag}(g_T)\, M \,\operatorname{diag}(1/\sin\phi_{d(e)}), \tag{4.1}$$

Both diagonal factors are invertible, so this block has rank $\operatorname{rank} M$. For every zero u and vertex i with $u_i > 0$, varying that coordinate slightly in both directions preserves nonnegativity. Copositivity and $u^{\mathsf T} S u = 0$ therefore give $(Su)_i = 0$. If $u_i = 0$, the following equation is trivial. Hence

$$u_i^2 = -\sum_{j \ne i} S_{ij} u_i u_j. \tag{4.2}$$

Each diagonal column of H is a fixed linear combination of its off-diagonal columns. It adds no rank. $\square$

Lemma 4.2 (cyclic incidence rank)**.** *For every odd $n = 2m + 1 \ge 5$,*

$$\operatorname{rank} M = \binom{n}{2} - n = \frac{n(n-3)}{2}. \tag{4.3}$$

Proof. Because n is odd, every unordered pair has a unique form $\{i, i + k\}$ with $1 \le k \le m$. Label it by (i, k) and write $x_{i,k}$ for its coordinate. For a surrounding triple with directed cyclic gaps $k, \ell, r \le m$ and $k + \ell + r = n$, the equation $Mx = 0$ is

$$x_{i,k} + x_{i+k,\ell} + x_{i+k+\ell,r} = 0. \tag{4.4}$$

Over $\mathbb{C}$, the functions $i \mapsto z^i$ for $z^n = 1$ form a basis of functions on $\mathbb{Z}/n\mathbb{Z}$. Thus every array x has the Fourier expansion

$$x_{i,k} = \sum_{z^n=1} z^i f_k(z), \qquad f_k(z) = \frac{1}{n} \sum_{i=0}^{n-1} z^{-i} x_{i,k}.$$

Substitution into (4.4) and comparison of the independent functions $i \mapsto z^i$ show that the complex kernel splits into **modes** $x_{i,k} = z^i f_k$, one for each z. The equations for a mode are

$$f_k + z^k f_\ell + z^{k+\ell} f_r = 0 \qquad (k + \ell + r = n;\ 1 \le k, \ell, r \le m). \tag{4.5}$$

For $z \neq 1$, subtract the equations for $(k, m+1-k, m)$ and $(k, m, m+1-k)$; the result is (4.6).

$$(1 - z^m) f_{m+1-k} = (1 - z^{m+1-k}) f_m. \tag{4.6}$$

Since $\gcd(m, n) = 1$, $z^m \neq 1$. As k varies, (4.6) fixes every f_j as a multiple of $1 - z^j$. Conversely these values satisfy (4.5), whose left side telescopes to $1 - z^n = 0$. Thus the kernel in each nontrivial mode has dimension one.

For $z = 1$, compare the equations for $(k, m+1-k, m)$ and $(k+1, m+1-k, m-1)$, $1 \leq k \leq m-1$. They give $f_{k+1} - f_k = f_m - f_{m-1}$, so $f_k = Ak + B$. Every equation (4.5) then reduces to $nA + 3B = 0$. This mode also has dimension one. There are n values of z, so $\dim_{\mathbb{C}} \ker M = n$. The matrix M is real, hence its real nullity is n as well. With $\binom{n}{2}$ columns, (4.3) follows. $\square$

By Proposition 3.4, Lemmas 2.1, 4.1, and 4.2, the exposed face $F(S) \subseteq \mathrm{CP}_n$ has maximum cp-rank exactly $n(n-3)/2$. This proves the odd case of the lower bound in Theorem 1.1.

5 One-coordinate restriction and even order

Let $N = 2m \geq 6$, let $n = N + 1 = 2m + 1$, and form the odd-order matrix S in (3.2). Delete coordinate 0 to obtain an $N \times N$ principal submatrix S'. It is copositive because a nonnegative vector on the surviving coordinates extends by zero. Such a vector is a zero of S' exactly when its zero extension is a zero of S. Proposition 3.4 therefore says that the zero rays of S' are precisely the surrounding-triple rays avoiding 0.

Let M' be the submatrix of M with rows for surrounding triples avoiding 0 and columns for pairs of surviving vertices. The rank transfer in Lemma 4.1 applies to this sublist: equation (4.1) restricts to its rows and surviving pairs, and (4.2) uses only surviving coordinates. We calculate $\operatorname{rank} M'$.

Proposition 5.1 (restriction rank). *For every $m \geq 3$,*

$$\operatorname{rank} M' = \binom{2m}{2} - 2m - 1 = \frac{N(N-3)}{2} - 1. \tag{5.1}$$

Proof. Divide the columns of the full odd-order matrix M into P_0, the pairs avoiding 0, and $P_\bullet$, the pairs through 0. Divide its rows into those avoiding 0, whose P_0 block is M', and the rows R_0 of surrounding triples through 0. Write U for the row space of M' padded by zeros on $P_\bullet$, and W for the row space of R_0. Since $\operatorname{rowspace} M = U + W$,

$$\operatorname{rank} M' = \operatorname{rank} M - \operatorname{rank} R_0 + \dim(U \cap W). \tag{5.2}$$

A triple $\{0, j, k\}$ with $1 \leq j < k \leq 2m$ is surrounding exactly when $j \leq m < k$ and $k - j \leq m$. Encode these triples by the edges of a bipartite graph B with left vertices $1, \ldots, m$ and right vertices $m+1, \ldots, 2m$, where j is adjacent to $m+1, \ldots, j+m$. Write $E(B)$ for its edge set. The graph is connected and has

$$|E(B)| = \sum_{j=1}^{m} j = \frac{m(m+1)}{2}. \tag{5.3}$$

Each row of R_0 contains the P_0 coordinate indexed by its pair $\{j, k\}$, which occurs in no other row of R_0. Thus its rows are independent and $\operatorname{rank} R_0 = |E(B)|$. For a pair $\{a, b\}$, let b_{ab} be its unit coordinate vector in the space indexed by all unordered pairs. An element of W is

$\sum_{jk\in E(B)} \lambda_{jk}(b_{0j}+b_{0k}+b_{jk})$. Its $P_\bullet$ coordinates vanish precisely when $\sum_{f\ni v}\lambda_f = 0$ at each vertex v of B. Define $Z(B) = \{\lambda \in \mathbb{R}^{E(B)} : \sum_{f\ni v}\lambda_f = 0$ for every vertex $v\}$, and use the same definition for subgraphs G of B. For an edge weighting $\lambda \in \mathbb{R}^{E(G)}$, its support is $\{e \in E(G) : \lambda_e \neq 0\}$. Orient each edge of G from left to right, and let D_G be its vertex-edge matrix, with -1 at the left endpoint, $+1$ at the right endpoint, and zero elsewhere. The equations $D_G\lambda = 0$ differ from the defining vertex-sum equations only by signs, so $Z(G) = \ker D_G$ is the real cycle space of G. The map $\pi(\lambda) = \sum_{jk\in E(B)} \lambda_{jk}b_{jk}$ is injective, so $\dim(U \cap W)$ equals the dimension of the $\lambda \in Z(B)$ for which $\pi(\lambda) \in U$.

The edge $e_* = \{m, m+1\}$ belongs to B but its column in M' is zero. Indeed no surrounding triple avoiding 0 contains both m and $m+1$: a third vertex on either complementary arc makes one cyclic gap exceed m. Hence $\pi(\lambda) \in U$ implies $\lambda_{e_*} = 0$. Let $B-e_*$ be the graph obtained by deleting this edge, and regard $Z(B-e_*)$ as a subspace of $\mathbb{R}^{E(B)}$ by assigning zero to its e_* coordinate. This gives the upper inclusion

$$\{\lambda \in Z(B) : \pi(\lambda) \in U\} \subseteq Z(B - e_*), \tag{5.4}$$

We prove the reverse inclusion by four-cycles. For any cycle in a bipartite graph, its **cycle vector** assigns alternating $+1$ and -1 to successive edges and zero to all other edges. A four-cycle in B has vertices $j < j' \le m < k < k'$ with $k' - j \le m$ and alternating pair-coordinate vector, the image of its cycle vector under π, $v = b_{jk} - b_{j'k} + b_{j'k'} - b_{jk'}$. It avoids e_* exactly when $j' \le m-1$ or $k \ge m+2$. In the first case set $\ell = j' + m + 1 > k'$; in the second set $\ell = k - m - 1 < j$. The defining inequalities of B show in either case that all four triples $\{j, k, \ell\}$, $\{j', k, \ell\}$, $\{j', k', \ell\}$, and $\{j, k', \ell\}$ are surrounding and avoid 0. Indeed, for $x \in \{j, j'\}$ and $y \in \{k, k'\}$ the cyclic gaps of $\{x, y, \ell\}$ are, in the first case, $y - x \le m$, $\ell - y \le m$, $n - \ell + x = m + x - j' \le m$; in the second case they are $x - \ell \le m$, $y - x \le m$, $n - y + \ell = m + k - y \le m$. If T_{xyz} denotes the incidence row of $\{x, y, z\}$, cancellation of the pairs through ℓ gives

$$T_{jk\ell} - T_{j'k\ell} + T_{j'k'\ell} - T_{jk'\ell} = v. \tag{5.5}$$

Thus every four-cycle vector of $B - e_*$ maps into U.

These four-cycles span $Z(B - e_*)$. For any nonzero λ in this space, a vertex incident with exactly one edge in its support would violate the vertex-sum equation. Thus every vertex incident with a support edge has degree at least two there, and the finite support subgraph contains a cycle. Subtract a multiple of that cycle's vector to eliminate one support edge, without adding edges to the support; finitely many repetitions show that cycle vectors span the space. Now consider a cycle C of length at least six. Choose its largest left vertex j_* other than m. Every right vertex k on C has a cycle-neighbor $j_a \neq m$, so $k \le j_a + m \le j_* + m$. Thus j_* is adjacent in $B - e_*$ to every right vertex of C; this edge cannot be e_*, since $j_* \neq m$. There are at least three right vertices but only two cycle-neighbors of j_*, so C has a chord, an edge joining two nonconsecutive vertices of C. It divides C into two shorter cycles. Choose the signs of their alternating cycle vectors so their chord entries cancel; their sum is the cycle vector of C. Repeating this split expresses that vector as a sum of four-cycle vectors. Also, $B - e_*$ remains connected when $m \ge 3$: m meets every right vertex except $m+1$, the latter meets every left vertex except m, and $m, m+1$ are joined by the path $m, (m+2), (m-1), (m+1)$. For $G = B - e_*$, the equation $D_G^{\mathsf{T}}c = 0$ makes the vertex labels c equal across every edge. Connectedness makes c constant, so $\operatorname{rank} D_G = 2m - 1$ and

$$\dim Z(B - e_*) = (|E(B)| - 1) - 2m + 1 = \frac{m(m-3)}{2}. \tag{5.6}$$

Equations (5.4)–(5.6) show that $\dim(U \cap W) = m(m-3)/2$. Substitute this, (5.3), and $\operatorname{rank} M = \binom{2m+1}{2} - (2m+1)$ from Lemma 4.2 into (5.2). The result is $\operatorname{rank} M' = 2m^2 - 3m - 1$, which is (5.1). $\square$

Proof of Theorem 1.1. The odd lower bound follows after Lemma 4.2. For even $n = N \geq 6$, the rank transfer of Lemma 4.1 applies to the surviving zero rays, so Proposition 5.1 and Lemma 2.1 give a matrix in the exposed face $F(S') \subseteq \mathrm{CP}_N$ of cp-rank $N(N-3)/2 - 1$. The upper bound is [2, Corollary 5.2]. Subtracting the two expressions in (1.2) gives $2n - 4 + \varepsilon_n$. $\square$

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