A Conjecture on the Permanent of Positive Semidefinite Matrices

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Abstract

We state the following conjecture of the author and prove it for $n \le 5$. Let $A = (a_{ij})$ be a complex positive semidefinite $n \times n$ matrix with $n \ge 1$, and let χ_λ be an irreducible character of S_n of degree d_λ. For $\sigma \in S_n$, put $a_\sigma = \prod_{i=1}^n a_{i,\sigma(i)}$. The conjecture asserts that

$$\sum_{\sigma,\tau \in S_n} \chi_\lambda(\sigma\tau) a_\sigma a_\tau \le d_\lambda \big(\operatorname{per} A\big)^2.$$

We show that Soules' permanent-on-top conjecture implies this inequality, and that it in turn implies both Lieb's immanant dominance and Chollet's permanent inequality. We also settle the previously unresolved order-four case of Soules' conjecture.

Keywords. Permanent; positive semidefinite matrix; Schur power matrix; immanant; Chollet conjecture.
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1 Introduction

Let S_n be the permutations of $[n] = \{1, \dots, n\}$. For a complex $n \times n$ matrix $A = (a_{ij})$, put

$$a_\sigma(A) = \prod_{i=1}^n a_{i,\sigma(i)}, \qquad \operatorname{per} A = \sum_{\sigma \in S_n} a_\sigma(A). \tag{1.1}$$

Throughout the paper, positive semidefinite means Hermitian positive semidefinite, written $A \succeq 0$. For Hermitian matrices of the same order, $X \preceq Y$ means $Y - X \succeq 0$. We write $\lambda \vdash n$ when λ is a partition of n; it labels an irreducible character χ_λ of S_n with degree $d_\lambda = \chi_\lambda(e)$. Our conjecture is the following quadratic inequality.

Conjecture 1.1 (quadratic permanental dominance)**.** *For every $n \ge 1$, every $A \succeq 0$ of order n, and every $\lambda \vdash n$,*

$$\boxed{\sum_{\sigma,\tau \in S_n} \chi_\lambda(\sigma\tau) a_\sigma(A) a_\tau(A) \le d_\lambda \big(\operatorname{per} A\big)^2.} \tag{1.2}$$

The sum on the left is real and nonnegative, as Section 2 explains.

Soules proposed a stronger spectral conjecture in his 1966 dissertation [10]; his 1994 paper [3] developed the approach. The **Schur power matrix** of A is the $n! \times n!$ matrix, indexed by S_n, with entries

$$\Pi(A)_{\sigma,\tau} = \prod_{i=1}^n a_{\sigma(i),\tau(i)}. \tag{1.3}$$

For $A \succeq 0$, this matrix is positive semidefinite and has $\operatorname{per} A$ as an eigenvalue. The **permanent-on-top conjecture** is that this is its largest eigenvalue. Bapat and Sunder [2] proved it for $n \leq 3$. Shchesnovich [4] gave a complex order-five counterexample. Its direct sum with I_{n-5} for $n > 5$ preserves the permanent and embeds the order-five Schur power matrix as a principal submatrix. Thus the conjecture fails for every $n \geq 5$. The order-four case remained open [8]. We settle it in Theorem 3.1.

Lieb's **permanental-dominance conjecture** [1] concerns the immanant

$$\operatorname{Imm}_\lambda(A) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) a_\sigma(A). \tag{1.4}$$

It asserts $\operatorname{Imm}_\lambda(A) \leq d_\lambda \operatorname{per} A$ for every $A \succeq 0$ and $\lambda \vdash n$.

Chollet's conjecture [5] concerns the entrywise product $(A \circ B)_{ij} = a_{ij} b_{ij}$: for two positive semidefinite matrices of the same order, it asserts

$$\operatorname{per}(A \circ B) \leq \operatorname{per} A \operatorname{per} B. \tag{1.5}$$

Pate established Lieb's conjecture for all irreducible S_n immanants through order 13 [11], and a recent preprint of Li claims the same through order 15 [15]. For Chollet's conjecture, Hutchinson proved the real order-four case [12], Rodtes the complex case [13], and a recent unrefereed preprint of Li, Zhao, and Luo claims the inequality through order six [14].

Our main result is Theorem 4.1: inequality (1.2) holds for every complex positive semidefinite matrix through order five and every partition of its order. For $n \leq 4$, this follows from our new order-four Soules theorem, the known cases through order three, and the implication theorem in the following section. Theorem 4.1 also yields alternative proofs of the order-five cases of Lieb's and Chollet's inequalities. Theorem 3.1 gives another proof of Chollet's complex order-four case.

The order-four proof is computer assisted: its finite polynomial identities and positivity certificates are checked with exact arithmetic. At order five, two character blocks have short analytic proofs; three further blocks use exact rational sums of squares. Section 5 identifies the proof data and replay programs.

2 The implication theorem

Theorem 2.1 (implications, at each fixed order)**.** *For all complex positive semidefinite matrices of a fixed order n,*

$$\begin{array}{c}\textit{Soules' permanent-on-top}\\ \textit{conjecture}\end{array} \Longrightarrow \textit{Conjecture 1.1} \Longrightarrow \begin{cases}\textit{Lieb's immanant inequality,}\\ \textit{Chollet's inequality.}\end{cases} \tag{2.1}$$

To prove the theorem, choose a unitary irreducible representation ρ_λ with character χ_λ and dimension d_λ, and define its Fourier block by

$$F_\lambda(A) = \sum_{\sigma \in S_n} a_\sigma(A) \rho_\lambda(\sigma). \tag{2.2}$$

The choice of equivalent representation does not affect the eigenvalues. The trivial block is the scalar $\operatorname{per} A$; the sign block is $\det A$. We record the positivity and normalization used throughout.

Lemma 2.2. *If $A \succeq 0$, then every $F_\lambda(A)$ is Hermitian positive semidefinite. Moreover,*

$$\operatorname{tr} F_\lambda(A)^2 = \sum_{\sigma,\tau \in S_n} \chi_\lambda(\sigma\tau) a_\sigma(A) a_\tau(A), \tag{2.3}$$

and the spectrum of $\Pi(A)$ consists of the spectrum of each $F_\lambda(A)$ repeated d_λ times.

Proof. Write A as the Gram matrix of vectors $v_1, \ldots, v_n$. Equation (1.3) is the Gram matrix of the tensors $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$. Reindexing the product in (1.3) gives $\Pi(A)_{\sigma,\tau} = a_{\tau\sigma^{-1}}(A)$. Choose real orthogonal models for the irreducible representations of S_n. The left-regular decomposition gives the unitary similarity

$$\Pi(A) \simeq \bigoplus_{\lambda \vdash n} F_\lambda(A^T) \otimes I_{d_\lambda}, \qquad F_\lambda(A^T) = \overline{F_\lambda(A)}.$$

The second identity uses that A is Hermitian and the representation matrices are real. Also $a_{\sigma^{-1}}(A) = \overline{a_\sigma(A)}$, so $F_\lambda(A)$ is Hermitian. It has the same spectrum as its complex conjugate, a positive semidefinite block of $\Pi(A)$, and is therefore positive semidefinite. Expanding the square in (2.2) and taking its trace proves (2.3). $\square$

Proof of Theorem 2.1. **Soules' permanent-on-top conjecture $\Longrightarrow$ Conjecture 1.1.** Under Soules' conjecture, Lemma 2.2 puts every eigenvalue of $F_\lambda(A)$ in $[0, \operatorname{per} A]$. Squaring and summing gives $\operatorname{tr} F_\lambda(A)^2 \le d_\lambda (\operatorname{per} A)^2$, which is (1.2).

Conjecture 1.1 $\Longrightarrow$ Lieb's immanant inequality. Assuming (1.2), Cauchy–Schwarz on the eigenvalues gives

$$\operatorname{Imm}_\lambda(A) = \operatorname{tr} F_\lambda(A) \le \sqrt{d_\lambda \operatorname{tr} F_\lambda(A)^2} \le d_\lambda \operatorname{per} A. \tag{2.4}$$

Conjecture 1.1 $\Longrightarrow$ Chollet's inequality. Assuming (1.2), Fourier orthogonality gives the exact identity

$$n! \operatorname{per}(A \circ \overline{A}) = \sum_{\lambda \vdash n} d_\lambda \operatorname{tr} F_\lambda(A)^2. \tag{2.5}$$

Since $\sum_{\lambda \vdash n} d_\lambda^2 = n!$, inequality (1.2) bounds the right side by $n!(\operatorname{per} A)^2$. Thus $\operatorname{per}(A \circ \overline{A}) \le (\operatorname{per} A)^2$. For another positive semidefinite B, Cauchy–Schwarz on the vectors $(a_\sigma(A))_\sigma$ and $(a_\sigma(B))_\sigma$ yields

$$\operatorname{per}(A \circ B) \le |\operatorname{per}(A \circ B)| \le \sqrt{\operatorname{per}(A \circ \overline{A}) \operatorname{per}(B \circ \overline{B})} \le \operatorname{per} A \operatorname{per} B. \tag{2.6}$$

This proves Chollet's inequality and completes the theorem. $\square$

3 Soules' conjecture in order four

Theorem 3.1. *If $A \succeq 0$ is a complex 4×4 matrix, then $\Pi(A) \preceq (\operatorname{per} A) I_{24}$.*

Proof. The proof reduces to five finite, exact characteristic certificates. We give the reduction and the certificate checks explicitly enough to identify every mathematical obligation; the full integer data and verifier are listed in Section 5.

For $\lambda \vdash 4$ put $\Delta_\lambda = (\operatorname{per} A) I_{d_\lambda} - F_\lambda(A)$. The trivial block has $\Delta = 0$. For the sign block, the classical permanent diagonal-product inequality [6] and Hadamard's determinant inequality give $0 \le \det A \le \prod_i a_{ii} \le \operatorname{per} A$. For the other three blocks we use the elementary-symmetric functions $e_k(\Delta)$ of their eigenvalues. The known order-four cases of Lieb's normalized immanant inequality [7, 11] give

$$e_1(\Delta_\lambda) = d_\lambda \operatorname{per} A - \operatorname{Imm}_\lambda(A) \ge 0. \tag{3.1}$$

For a Hermitian 2×2 matrix, $e_1, e_2 \ge 0$ imply positive semidefiniteness; for a Hermitian 3×3 matrix, the same is true of $e_1, e_2, e_3 \ge 0$. Indeed, if an eigenvalue were negative, the polynomial $\det(tI + \Delta) = t^d + e_1 t^{d-1} + \cdots + e_d$ would vanish at some $t > 0$, which its nonnegative coefficients forbid. The remaining checks are therefore these five signs:

λ	d_λ	Characteristic signs	Exact certificate
$(2,2)$	2	$e_2 \geq 0$	rational matrix sum of squares
$(3,1)$	3	$e_2 \geq 0$, $e_3 \geq 0$	rational matrix sum of squares; ten principal minors
$(2,1,1)$	3	$e_2 \geq 0$, $e_3 \geq 0$	six parity minors; ten principal minors

We next describe the domain on which the certificates are applied. Positive diagonal congruence scales every Fourier block and $\operatorname{per} A$ by the same positive factor, so we may work with correlation matrices. If a correlation matrix B of order four is singular, then $\operatorname{rank} B \leq 3$ and $\operatorname{tr} B^2 \geq (\operatorname{tr} B)^2/3 = 16/3$. Consequently $\sum_{i<j} |b_{ij}|^2 \geq 2/3$, and some pair has $|b_{ij}| \geq 1/3$. Relabel and apply a diagonal unitary congruence so that this pair is $(1,2)$ with inner product $\rho \geq 1/3$. Generic rank-three frames admit, after unitary changes and positive rescaling, the triangular chart

$$v_1 = (1,0,0), \quad v_2 = (a,1,0), \quad v_3 = (b, c+id, 1), \quad v_4 = x \in \mathbb{C}^3, \qquad a^2 = \frac{\rho^2}{1-\rho^2} \geq \frac{1}{8}, \tag{3.2}$$

with a,b,c,d real. For the $(3,1)$ determinant certificate it is more convenient to use the equivalent symmetric-pair chart

$$v_1 = (u,1,0), \quad v_2 = (u,-1,0), \quad v_3 = (b, c+id, 1), \quad v_4 = x, \qquad u^2 = \frac{1+\rho}{1-\rho} \geq 2. \tag{3.3}$$

In either chart, a phase change of v_3, compensated by a unitary change of the third Gram coordinate, makes b real. Positive scaling of x is harmless for the homogeneous sign checks, so x needs no normalization.

The sign checks in the table are exact assertions on the respective chart domains. To state how they are certified, let $z_k(x)$ list the $\binom{k+2}{2}$ holomorphic monomials of degree k in the three coordinates of x, and let $\mathbf{t}$ denote the real variables of the chosen chart. Each $a_\sigma(B)$ has bidegree $(1,1)$ in $x, \bar{x}$; the real-valued characteristic expression $e_k(\Delta_\lambda(x))$ is therefore a Hermitian form of bidegree (k,k). It has a unique Hermitian coefficient matrix in the z_k basis. For each entry in the table, the verifier constructs this matrix $C_{\lambda,k}$ and checks the complete coefficient identity

$$e_k(\Delta_\lambda(x)) = z_k(x)^* C_{\lambda,k}(\mathbf{t}) z_k(x). \tag{3.4}$$

The condition $C_{\lambda,k} \succeq 0$ is sufficient for the required nonnegativity.

The two rational matrix sums of squares prove their 6×6 matrices positive semidefinite after multiplication by $1 + a^2 > 0$. The $(2,1,1)$ quadratic matrix has six exact nonnegative principal minors. For each determinant matrix, of size 10×10, all ten leading principal minors have exact sign certificates: in the triangular chart the last sign uses $a^2 \geq 1/8$, and in the symmetric chart it uses $u^2 \geq 2$. The final near-column minor is an exact polynomial of the form $R(a^2, b^2, c^2, d^2) + abc\, Q(a^2, b^2, c^2, d^2)$; its certificate proves $R \geq 0$ and $R^2 - a^2b^2c^2Q^2 \geq 0$ on the stated domain. The final standard-hook minor is split exactly into 24 homogeneous layers in b^2, c^2, d^2, with polynomial factorization and rational chart certificates covering the nonnegative orthant. All factorizations, coefficient comparisons, rational inequalities, and domain covers are replayed in integer arithmetic. Each certified leading-minor polynomial is nonzero and nonnegative on the chart domain, hence positive on a dense open subset. The finite intersection of these subsets is dense and open, so Sylvester's criterion applies there; continuity handles the chart boundaries and nongeneric frames.

It remains to pass from singular to arbitrary correlation matrices. If $A \neq I$ is positive definite, let $m = \lambda_{\min}(A)$, $t = 1 - m$, and $B = (A - mI)/t$. Then $0 < t < 1$, B is a singular correlation matrix, and $A=(1-t)I+tB$. Expansion of the diagonal choices gives

$$\Delta_\lambda(A)=\sum_{S\subseteq[4]} t^{|S|}(1-t)^{4-|S|}\left[\operatorname{per}(B[S])I-\sum_{\sigma\in S_S} a_\sigma(B[S])\rho_\lambda(\sigma)\right], \tag{3.5}$$

where S_S fixes the complement of S and $\operatorname{per}(B[\varnothing])=1$. The $S=[4]$ summand is positive semidefinite by the singular certificate. For a proper S, restrict ρ_λ to S_S; each summand is a direct sum of blocks of order at most three, where Bapat and Sunder's theorem [2] applies. Every term in (3.5) is therefore positive semidefinite. The identity matrix is immediate. Positive diagonal congruence covers every positive semidefinite matrix with positive diagonal; continuity covers zero diagonal entries. Lemma 2.2 now gives $\Pi(A)\preceq(\operatorname{per}A)I_{24}$, proving Theorem 3.1. □

4 Proof of Conjecture 1.1 through order five

Theorem 4.1. *Inequality* (1.2) *holds for every $n\le 5$, every complex positive semidefinite $n\times n$ matrix, and every $\lambda\vdash n$.*

Proof. For $n\le 4$ this follows at once from Bapat and Sunder's theorem [2], Theorem 3.1, and the first implication in Theorem 2.1. We therefore treat $n=5$. Positive diagonal congruence reduces matrices with positive diagonal to correlation matrices, and continuity covers zero diagonal entries. The trivial block is equality, while for the sign block $F_{(1^5)}(A)=\det A\le\operatorname{per}A$. Five nontrivial blocks remain.

We first handle $(4,1)$. For a positive semidefinite 4×4 correlation matrix B, let $\operatorname{padj}(B)_{ij}=\operatorname{per}B(i|j)$, where $B(i|j)$ deletes row i and column j. The permanental adjugate is positive semidefinite: it is the Gram matrix of the symmetrized tensors formed by deleting one Gram vector. We claim

$$\lambda_{\max}(\operatorname{padj}B)\le 2\operatorname{per}B. \tag{4.1}$$

Indeed, isolate row and column 4 and write $B=\left(\begin{smallmatrix}C & c\\ c^* & 1\end{smallmatrix}\right)$, $R=\operatorname{padj}(C)^T$, and $q=c^*Rc$. Then $R\succeq 0$, $\operatorname{per}B=\operatorname{per}C+q$, and expansion of the permanental cofactors along the isolated row shows that its absolute row sum is $\operatorname{per}C+\|Rc\|_1$. A phase choice puts the three off-diagonal moduli of C in $a,b,t\in[0,1]$ and its remaining phase at θ; with $u=a^2+b^2+t^2$ one has

$$\begin{aligned}\operatorname{per}C&=1+u+2abt\cos\theta\ge\max(1,2u),\\ \omega^*R\omega&\le 3+2\sqrt{3u}+3u\le 8\operatorname{per}C\quad(|\omega_j|=1).\end{aligned} \tag{4.2}$$

Indeed $\operatorname{per}C=\det C+2u\ge 2u$, while $2abt\le a^2+b^2$ gives $\operatorname{per}C\ge 1$. If $u\le 1/2$, the middle expression in (4.2) is at most $9/2+\sqrt{6}<8$; if $u\ge 1/2$, its difference from $16u$ is $13u-2\sqrt{3u}-3\ge 7/2-\sqrt{6}>0$. This proves the last bound in (4.2). Duality of ℓ^1 and ℓ^∞, followed by Cauchy–Schwarz in the R-seminorm, yields $\|Rc\|_1\le\sqrt{8(\operatorname{per}C)q}\le\operatorname{per}C+2q$. Thus every absolute row sum of $\operatorname{padj}B$ is at most $2\operatorname{per}B$, proving (4.1).

Now let A be a 5×5 correlation matrix and set $\Gamma_{ij}=a_{ij}\operatorname{per}A(i|j)$. The natural permutation representation is the sum of the trivial and $(4,1)$ representations; its Fourier matrix is Γ up to transpose. Hence

$$\|\Gamma\|_F^2=(\operatorname{per}A)^2+\operatorname{tr}F_{(4,1)}(A)^2. \tag{4.3}$$

For any chosen row, write $A=\left(\begin{smallmatrix}B & c\\ c^* & 1\end{smallmatrix}\right)$ after relabeling, with $R=\operatorname{padj}(B)^T$ and $q=c^*Rc$, so $\operatorname{per}A=\operatorname{per}B+q$. Because $|c_j|\le 1$, (4.1) gives

$$\sum_{j\ne 5}|\Gamma_{5j}|^2\le c^*R^2c\le 2(\operatorname{per}B)q,\qquad |\Gamma_{55}|^2=(\operatorname{per}B)^2. \tag{4.4}$$

Each of the five rows has squared norm at most $(\operatorname{per} B + q)^2 = (\operatorname{per} A)^2$. Equations (4.3) and (4.4) give $\operatorname{tr} F_{(4,1)}(A)^2 \le 4(\operatorname{per} A)^2$.

For $(2,1,1,1)$, consider first a singular 5×5 correlation matrix B. The sign-twisted natural representation has Fourier matrix $K_{ij} = (-1)^{i+j} b_{ij} \det B(i|j)$, up to transpose, and decomposes as the sign representation plus $(2,1,1,1)$. If $\operatorname{rank} B \le 3$, all fourth-order cofactors vanish and the desired block is zero. If $\operatorname{rank} B = 4$, then $\det B = 0$ and

$$\operatorname{tr} F_{(2,1,1,1)}(B) = \operatorname{tr} K = e_4(\operatorname{spec} B) \le (5/4)^4. \tag{4.5}$$

The last inequality is AM–GM on the four positive eigenvalues, whose sum is five. Grone and Pierce's correlation inequality [9] gives $\operatorname{per} B \ge \|B\|_F^2/5 \ge 5/4$ on this singular boundary. For a positive semidefinite 4×4 block, its Frobenius norm is at most its trace, so (4.5) implies

$$\frac{\sqrt{\operatorname{tr} F_{(2,1,1,1)}(B)^2/4}}{\operatorname{per} B} \le \frac{(5/4)^4}{2(5/4)} = \frac{125}{128} < 1. \tag{4.6}$$

For a nonsingular correlation matrix use the order-five analogue of (3.5): $A = (1-t)I + tB$, with weights $t^{|S|}(1-t)^{5-|S|}$ over $S \subseteq [5]$. In every proper-subset term, Young restriction yields blocks of order at most four. Bapat and Sunder [2] bound those of order at most three in operator norm by their permanents, and Theorem 3.1 bounds those of order four.

Consequently the full 4×4 restricted Fourier matrix $X_S = \sum_{\sigma \in S_S} a_\sigma(B[S])\rho_{(2,1,1,1)}(\sigma)$ satisfies $\|X_S\|_F \le 2 \operatorname{per}(B[S])$. The full-set term obeys (4.6), and the same weighted subset expansion holds for $\operatorname{per} A$. The triangle inequality for normalized Frobenius norm now gives $\sqrt{\operatorname{tr} F_{(2,1,1,1)}(A)^2/4} \le \operatorname{per} A$. The singular correlation case was treated above, so this covers all A.

The remaining three partitions are $(3,2)$, $(3,1,1)$, and $(2,2,1)$, of dimensions $5, 6, 5$. For each of them put

$$R_\lambda(A) = d_\lambda(\operatorname{per} A)^2 - \operatorname{tr} F_\lambda(A)^2 = \sum_{\sigma,\tau \in S_5} (d_\lambda - \chi_\lambda(\sigma\tau)) a_\sigma(A) a_\tau(A). \tag{4.7}$$

The exact certificates supplied with this paper have the following form:

$$R_\lambda(A) = \frac{1}{D_\lambda} \sum_{b=1}^{38} \mathcal{R}\left(\prod_{S \in \mathcal{B}_b} \det A[S] \; z_b(A)^* Q_b z_b(A) \right). \tag{4.8}$$

Here D_λ is a positive integer, $\mathcal{B}_b$ is an explicit list of principal index sets, z_b is an explicit vector of monomials in Gram entries, Q_b is a symmetric integer matrix, and $\mathcal{R}$ averages simultaneous relabeling by S_5 and entrywise conjugation. Every determinant in (4.8) is nonnegative on $A \succeq 0$. Every Q_b is positive definite: the certificate gives an integer congruence whose image has positive diagonal and is strictly diagonally dominant, hence positive definite. This forces the congruence matrix to be invertible and therefore $Q_b \succ 0$. Its verifier checks these integer margins and the complete coefficient identity (4.8). There are 6,210 balanced degree-ten monomials, grouped into 82 relabeling/conjugation orbits; explicit averaging makes equality of the 82 exact orbit coefficients equality of all 6,210 coefficients. The exact denominators are

λ	D_λ	Positive matrices
$(3,2)$	28022233339265024	38
$(3,1,1)$	28022233339265024	38
$(2,2,1)$	14011116669632512	38

Thus (4.8) proves $R_\lambda(A) \ge 0$ for all three remaining blocks, including singular matrices. All seven partitions of five have now been covered, and Theorem 4.1 follows. $\square$

5 Supplementary material

The supplement guide[1] identifies the exact certificates, full order-four input tree, replay programs, completed receipt, and SHA-256 digests. All paths below are relative to the supplement root. For order four, `code/audit_pstar_program_soules.py` reconstructs the five characteristic matrices from the Gram charts, checks the character identities on complete degree-controlled grids, verifies both rational matrix sums of squares and all characteristic minors by exact polynomial division, then checks every final sign certificate. Its six-component completion receipt is `reports/soules/summary.json`. The order-five function `code/certify_n5_p2_minor_sos.py:verify` checks (4.8) independently for each of the three supplied gzip JSON certificates. Numerical optimization was used to find candidates, but no floating-point acceptance step enters either proof.

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