

Spectrally Simple Spectra without Irreducible Nonnegative Realizations

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Abstract

We give a counterexample to a conjecture of Johnson, Marijuán, and Pisonero: we show that the spectrum $\{1, 1/2, \pm i/\sqrt{3}\}$ is realizable and spectrally simple, but has no irreducible nonnegative realization. More generally, we prove their nonexistence conjecture for an order-four boundary family.

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1 Introduction and main results

For an integer $n \geq 1$, let I_n denote the $n \times n$ identity matrix. The **spectrum** of a real $n \times n$ matrix A is the list of its eigenvalues, counted with algebraic multiplicity. A list σ of n complex numbers is **realizable** if it is the spectrum of an entrywise nonnegative real matrix. We write $A \geq 0$ when every entry of A is nonnegative. Such a matrix $A = (a_{ij})$ is **irreducible** if, for every nonempty proper subset S of $\{1, \dots, n\}$, some $i \in S$ and $j \notin S$ satisfy $a_{ij} > 0$. A spectrum is **irreducibly realizable** if it has an irreducible nonnegative realization.

For a realizable spectrum σ , its **Perron root** is $\rho(\sigma) = \max_{\lambda \in \sigma} |\lambda|$, which, by the Perron–Frobenius theorem, is a nonnegative real member of σ . Its **spectral circle** is $\{z \in \mathbb{C} : |z| = \rho(\sigma)\}$. Following Johnson, Marijuán, and Pisonero [1], we call σ **spectrally simple** if its Perron root occurs once and every other member has strictly smaller modulus. For a matrix A , write $\rho(A)$ for the maximum modulus of its eigenvalues.

Johnson, Marijuán, and Pisonero [1, p. 3] conjectured that every realizable, spectrally simple spectrum is irreducibly realizable. They proved the assertion in orders below four and for real spectra of order four [1, Sections 8–10]. In their analysis of nonreal spectra of order four, they also conjectured that a particular family has no irreducible realizations [1, p. 12]. Our main result proves the latter conjecture and disproves the former.

Theorem 1.1. *Let $a, b, c, d \in \mathbb{R}$ satisfy $a > b$, $a > 0$, and $d > 0$. Put $s = a + 2c$, and suppose*

$$s > 0, \quad a - c = \sqrt{3}d, \quad b > \frac{s}{3}. \quad (1.1)$$

Then $\sigma = \{a, b, c + di, c - di\}$ is realizable and spectrally simple, but is not irreducibly realizable.

Corollary 1.2. *The spectrum*

$$\left\{1, \frac{1}{2}, \frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}\right\} \quad (1.2)$$

is realizable and spectrally simple, but has no irreducible nonnegative realization.

The proof uses an eigenvector for the smaller real eigenvalue b . Its positive and negative coordinates force a strict lower bound on the sum of products $a_{ij}a_{ji}$. The coefficient of x^2 in the characteristic polynomial gives an incompatible upper bound. The argument includes eigenvectors with zero coordinates.

2 An eigenvector inequality

For a nonnegative 4×4 matrix $A = (a_{ij})$, put

$$l_i = a_{ii}, \quad q_{ij} = a_{ij}a_{ji} \quad (i < j), \quad Q = \sum_{i < j} q_{ij}. \quad (2.1)$$

Lemma 2.1. *Let $A \geq 0$ be an irreducible 4×4 matrix, and let $b < \rho(A)$ be a real eigenvalue. Suppose $0 \leq l_i \leq b$ for every i , with at most one $l_i = b$. Put $r_i = b - l_i$. After simultaneously permuting the rows and columns,*

$$Q > r_1r_2 + r_3r_4. \quad (2.2)$$

Proof. Choose a real eigenvector x with $Ax = bx$. The Perron–Frobenius theorem gives a real vector w whose entries are all positive and which satisfies $w^T A = \rho(A)w^T$. Since $(\rho(A) - b)w^T x = 0$ and $b < \rho(A)$, we have $w^T x = 0$. Because w has strictly positive entries, x has both positive and negative coordinates.

Suppose first that no coordinate is zero. A sign cannot occur just once: if x_i were the sole positive coordinate, the equation

$$r_i x_i = \sum_{j \neq i} a_{ij} x_j \leq 0$$

would force $r_i = 0$ and $a_{ij} = 0$ for all $j \neq i$, contrary to irreducibility. Replacing x by $-x$ treats a sole negative coordinate. We may therefore arrange $x_1, x_2 > 0$ and $x_3, x_4 < 0$. Write $u_i = x_i$ for $i = 1, 2$ and $v_i = -x_i$ for $i = 3, 4$. The first two rows give

$$\begin{aligned} a_{12}u_2 &= r_1u_1 + a_{13}v_3 + a_{14}v_4, \\ a_{21}u_1 &= r_2u_2 + a_{23}v_3 + a_{24}v_4. \end{aligned} \quad (2.3)$$

Both left sides are positive: if one vanished, its row would have no positive off-diagonal entry. Irreducibility supplies a positive entry from $\{1, 2\}$ to $\{3, 4\}$, so at least one of $a_{13}, a_{14}, a_{23}, a_{24}$ is positive. Since $v_3, v_4 > 0$, the sums

$$a_{13}v_3 + a_{14}v_4, \quad a_{23}v_3 + a_{24}v_4$$

in (2.3) are nonnegative, and at least one is positive. Consequently,

$$a_{12}u_2 \geq r_1u_1, \quad a_{21}u_1 \geq r_2u_2,$$

with at least one strict inequality. Both left sides are positive, as established above. Multiplying gives

$$q_{12}u_1u_2 = (a_{12}u_2)(a_{21}u_1) > r_1r_2u_1u_2.$$

If $r_1r_2 = 0$, the strict inequality follows directly from positivity of the left side. Dividing by $u_1u_2 > 0$ yields $q_{12} > r_1r_2$.

The same argument, using a positive entry from $\{3, 4\}$ to $\{1, 2\}$, gives $q_{34} > r_3r_4$. All remaining q_{ij} are nonnegative, so (2.2) follows from

$$Q \geq q_{12} + q_{34} > r_1r_2 + r_3r_4.$$

If x has two zero coordinates, its nonzero coordinates have opposite signs. After relabelling, suppose $x_1 > 0$, $x_2 < 0$, and $x_3 = x_4 = 0$. The first two row equations become

$$r_1 x_1 = a_{12} x_2, \quad r_2 x_2 = a_{21} x_1.$$

In the first equation, the left side is nonnegative and the right side is nonpositive; in the second, the signs are reversed. Thus both equations have both sides zero. Since $x_1, x_2 \neq 0$, we obtain $r_1 = r_2 = 0$, hence $l_1 = l_2 = b$, contradicting the assumption that at most one diagonal entry equals b . More than two zero coordinates cannot give both signs.

It remains to consider exactly one zero. After relabelling and, if necessary, replacing x by $-x$, assume $x_1, x_2 > 0$, $x_3 < 0$, and $x_4 = 0$. The third row gives $r_3 = 0$ and $a_{31} = a_{32} = 0$. Irreducibility forces $a_{34} > 0$. Applied to the subset $\{3, 4\}$, it also forces $a_{41} > 0$ or $a_{42} > 0$. The fourth row equation

$$a_{41} x_1 + a_{42} x_2 + a_{43} x_3 = 0$$

therefore implies $a_{43} > 0$. Hence $q_{34} > 0 = r_3 r_4$. The first two row equations, multiplied as above, give $q_{12} \geq r_1 r_2$. This again proves (2.2). $\square$

3 Proof of the main result

For a list $z = (z_1, z_2, z_3, z_4)$, define $e_2(z) = \sum_{i < j} z_i z_j$.

Proof of Theorem 1.1. The definition of s and (1.1) give

$$c = \frac{s - a}{2}, \quad d = \frac{3a - s}{2\sqrt{3}}, \quad 0 < s < 3a. \quad (3.1)$$

The nonnegative matrix

$$B = \frac{1}{3} \begin{pmatrix} s & 3a - s & 0 \\ 0 & s & 3a - s \\ 3a - s & 0 & s \end{pmatrix} \quad (3.2)$$

has characteristic polynomial $(x - s/3)^3 - (a - s/3)^3$ and eigenvalues $a, c + di, c - di$. Adjoin a first row and column to B , placing b in the top-left entry and zero in every other new entry. The resulting nonnegative matrix has characteristic polynomial $(x - b) \det(xI_3 - B)$ and realizes σ . Moreover,

$$c^2 + d^2 = a^2 - s \left(a - \frac{s}{3} \right) < a^2,$$

and $0 < b < a$, so σ is spectrally simple.

Suppose now that $A = (a_{ij})$ is an irreducible nonnegative realization. Put

$$l_i = a_{ii}, \quad q_{ij} = a_{ij} a_{ji} \quad (i < j), \quad Q = \sum_{i < j} q_{ij},$$

and write $l = (l_1, l_2, l_3, l_4)$. Write the characteristic polynomial as

$$\det(xI_4 - A) = x^4 + k_1 x^3 + k_2 x^2 + k_3 x + k_4.$$

The first two characteristic coefficients are k_1 , the coefficient of x^3 , and k_2 , the coefficient of x^2 . Computing them from the matrix entries and from the prescribed eigenvalues gives

$$k_1 = - \sum_i l_i = -(b + s), \quad k_2 = e_2(l) - Q = e_2(\sigma) = s \left(b + \frac{s}{3} \right). \quad (3.3)$$

Indeed, expanding the determinant gives $\sum_{i < j} (l_i l_j - a_{ij} a_{ji})$ as the coefficient of x^2 .

Let $m = \max_i l_i$. The other three diagonal entries have sum $b + s - m$; their pairwise products sum to at most $(b + s - m)^2/3$, with equality only when those entries are equal. Since $Q \geq 0$, (3.3) yields

$$s \left(b + \frac{s}{3} \right) \leq e_2(l) \leq m(b + s - m) + \frac{(b + s - m)^2}{3}. \quad (3.4)$$

The last expression, as a function of m , is strictly decreasing for $m > (b + s)/4$. Since $b > s/3$, we have $b > (b + s)/4$, and its value at $m = b$ is $s(b + s/3)$. Consequently $m \leq b$. If $m = b$, equality holds throughout (3.4), so $Q = 0$ and the other three diagonal entries are $s/3 < b$. Thus $0 \leq l_i \leq b$, with at most one $l_i = b$.

Lemma 2.1 applies, because $b < a = \rho(A)$. Put $r = (r_1, r_2, r_3, r_4)$, where $r_i = b - l_i$, and label the indices as in that lemma. Expanding $e_2(l) = e_2(b - r_1, \dots, b - r_4)$ in (3.3) gives

$$\sum_i r_i = 3b - s, \quad Q = e_2(r) - 3b^2 + 2bs - \frac{s^2}{3}. \quad (3.5)$$

Set $X = r_1 + r_2$ and $Z = r_3 + r_4$. Since $X + Z = 3b - s$, we have $XZ \leq (3b - s)^2/4$. It follows that

$$\begin{aligned} 0 < Q - r_1 r_2 - r_3 r_4 &= XZ - 3b^2 + 2bs - \frac{s^2}{3} \\ &\leq \frac{(3b - s)^2}{4} - 3b^2 + 2bs - \frac{s^2}{3} \\ &= -\frac{(3b - s)^2}{12} < 0. \end{aligned} \quad (3.6)$$

This contradiction proves that σ is not irreducibly realizable. $\square$

Proof of Corollary 1.2. Take $(a, b, c, d) = (1, 1/2, 0, 1/\sqrt{3})$. Then $s = 1$, and every hypothesis of Theorem 1.1 holds. The construction in (3.2) gives the explicit realization

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}.$$

Theorem 1.1 excludes every irreducible nonnegative realization. $\square$

Corollary 3.1. *The conjecture of Johnson, Marijuán, and Pisonero [1, p. 3] that every realizable, spectrally simple spectrum is irreducibly realizable is false.*

Proof. Corollary 1.2 gives a realizable, spectrally simple spectrum with no irreducible nonnegative realization. $\square$

Johnson, Marijuán, and Pisonero [1, Section 10, p. 12] also proposed the following family conjecture. Let $a, b, c, d \in \mathbb{R}$ satisfy $a > b$, $a > 0$, and $d > 0$. Suppose $\{a, c \pm di\}$ is realizable and

$$a + 2c > 0, \quad a - c = \sqrt{3}d, \quad b > \frac{a + 2c}{3}.$$

They conjectured that $\sigma = \{a, b, c \pm di\}$ has no irreducible nonnegative realization.

Corollary 3.2. *The family conjecture of Johnson, Marijuán, and Pisonero stated above is true.*

Proof. Put $s = a + 2c$. The parameters satisfy every hypothesis of Theorem 1.1, which excludes every irreducible nonnegative realization of σ . $\square$

Acknowledgments

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References

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