The Wang Transform Inequality

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Abstract

We prove the Wang transform inequality in every order: if n is a positive integer, A is a nonnegative $n \times n$ matrix whose row and column sums equal one, and J_n has every entry $1/n$, then

$$\operatorname{per} A \geq \operatorname{per}\left(\frac{nJ_n + A}{n+1}\right).$$

Equality holds exactly when $A = J_n$. We also obtain an explicit positive quadratic gap for every $n \geq 2$.

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1 Introduction and main results

For a positive integer n, let $[n] = \{1, \ldots, n\}$ and let S_n be its permutation group. The *permanent* of an $n \times n$ matrix $D = (d_{ij})$ is

$$\operatorname{per} D = \sum_{\pi \in S_n} \prod_{i=1}^{n} d_{i,\pi(i)}. \tag{1.1}$$

A real matrix is *doubly stochastic* if its entries are nonnegative and every row and column sum equals one. Write Ω_n for the set of such $n \times n$ matrices and J_n for the matrix with every entry $1/n$. Wang's transform is the map $W_n : \Omega_n \to \Omega_n$ defined by

$$W_n(A) = \frac{nJ_n + A}{n+1}. \tag{1.2}$$

Wang proposed inequality (1.3) [1]. Minc included it in his catalogue of open problems [2, Section 8.4] and restated it as Conjecture 4 in his 1983 survey [3, Section 5]. Cheon and Wanless retained it as Conjecture 4 in their later update [4]. Lih and Wang proved a stronger monotonicity result in order three [5]. Chang and Foregger independently proved order four [4, 7, 8]. Chang handled matrices outside a neighbourhood of J_n [6], and Hwang handled matrices that become a nontrivial block-diagonal direct sum after row and column permutations [4, 9].

Theorem 1.1. *For every positive integer n and every $A \in \Omega_n$, the following inequality holds, with equality if and only if $A = J_n$:*

$$\operatorname{per} A \geq \operatorname{per} W_n(A). \tag{1.3}$$

The proof compares a lower bound for $\operatorname{per} A$ with an upper bound for $\operatorname{per} W_n(A)$.

For a real matrix $X = (x_{ij})$, its Frobenius norm is defined by $\|X\|_F^2 = \sum_{i,j} x_{ij}^2$. Write $r^2 = \|A - J_n\|_F^2$, and put $\gamma_n = n!/n^n$ for every $n \geq 1$. For $n \geq 2$, define

$$C_n = \frac{\sum_{k=0}^{n} n!/k!}{(n+1)^n}, \quad a_n = \frac{1}{2n} + \frac{1}{3n^2}, \quad b_n = \frac{C_n/\gamma_n - 1}{n-1}. \tag{1.4}$$

Theorem 1.2. *For every integer $n \geq 2$ and every $A \in \Omega_n$,*

$$\operatorname{per} A \geq \gamma_n e^{a_n r^2}, \quad \operatorname{per} W_n(A) \leq \gamma_n (1 + b_n r^2). \tag{1.5}$$

We will show that $0 < b_n < 1/(2n) < a_n$ for every $n \geq 2$. The bounds in (1.5) then give the global gap

$$\begin{aligned} \operatorname{per} A - \operatorname{per} W_n(A) &\geq \gamma_n (e^{a_n r^2} - 1 - b_n r^2) \\ &\geq \gamma_n (a_n - b_n) r^2. \end{aligned} \tag{1.6}$$

The positive coefficient in the last bound is explicit; no optimality is claimed. We proved the lower estimate and the centered subpermanent bound used here in [10]. Section 2 states these inputs precisely. Section 3 derives the Wang-specific upper estimate, and Section 4 compares the constants and completes the proof.

2 Input estimates

We proved the lower estimate in (1.5) for every $n \geq 2$ and every $A \in \Omega_n$ in [10, Theorem 1.2, first inequality], including matrices on the boundary. We use that result without repeating its proof.

For a real $n \times n$ matrix X and row and column sets R, S of equal size, write $X[R, S]$ for the corresponding submatrix. Let

$$\sigma_s(X) = \sum_{\substack{R,S \subset [n] \\ |R| = |S| = s}} \operatorname{per} X[R, S], \qquad \sigma_0(X) = 1.$$

A matrix has *zero margins* if every row and column sum is zero. Let I_n be the identity matrix, let $Q = I_n - J_n$, and set $\|X\|_{\mathrm{op}} = \sup_{v \neq 0} \|Xv\|_2 / \|v\|_2$.

Lemma 2.1 (Centered subpermanents; [10], Lemma 3.1)**.** *For every integer $n \geq 2$, every real $n \times n$ matrix X with zero margins and $\|X\|_{\mathrm{op}} \leq 1$, and every $2 \leq s \leq n$,*

$$|\sigma_s(X)| \leq \frac{\sigma_s(Q)}{n-1} \|X\|_F^2. \tag{2.1}$$

We proved this estimate in [10, Lemma 3.1]. Its proof also establishes $\sigma_s(Q) \geq 0$ for $2 \leq s \leq n$ [10, equation (3.5)]. As verified just after that lemma in [10], if $A \in \Omega_n$ then $X = A - J_n$ has zero margins and $\|X\|_{\mathrm{op}} \leq 1$, so (2.1) applies.

3 The transformed upper bound

Let $X = A - J_n$. In [10, equation (4.1)] we proved the *centered expansion*, valid for every real t:

$$\operatorname{per}(J_n + tX) = \sum_{s=0}^{n} \frac{(n-s)!}{n^{n-s}} t^s \sigma_s(X). \tag{3.1}$$

Here $\sigma_1(X) = 0$ because the entries of X sum to zero, and the $s = 0$ term equals γ_n. Substitute $t = 1/(n+1)$ in (3.1), apply Lemma 2.1 term by term, and use $r^2 = \|X\|_F^2$:

$$\operatorname{per} W_n(A) \le \gamma_n + \frac{r^2}{n-1} \sum_{s=2}^{n} \frac{(n-s)!}{n^{n-s}(n+1)^s} \sigma_s(Q). \tag{3.2}$$

The sum is exactly $\operatorname{per} W_n(I_n) - \gamma_n$ by (3.1), since $I_n - J_n = Q$ and $\sigma_1(Q) = 0$. Let $\mathbf{1} = (1, \ldots, 1)^{\mathsf{T}}$. Then

$$\begin{aligned} W_n(I_n) &= \frac{\mathbf{1}\mathbf{1}^{\mathsf{T}} + I_n}{n+1}, \\ \operatorname{per}(\mathbf{1}\mathbf{1}^{\mathsf{T}} + I_n) &= \sum_{k=0}^{n} \binom{n}{k} (n-k)! = \sum_{k=0}^{n} \frac{n!}{k!}. \end{aligned} \tag{3.3}$$

The first sum counts the terms obtained by choosing identity entries in a subset of positions before taking the permanent of the remaining all-ones matrix. Hence $\operatorname{per} W_n(I_n) = C_n$, and (3.2) becomes

$$\operatorname{per} W_n(A) \le \gamma_n + \frac{C_n - \gamma_n}{n-1} r^2 = \gamma_n(1 + b_n r^2), \tag{3.4}$$

proving the second half of Theorem 1.2.

4 The constant comparison and final proof

4.1 Comparison of the constants

It remains to show that the two coefficients in (1.5) separate strictly. The sum in (3.3) is the integer $\lfloor en! \rfloor$: the omitted tail of $en! = \sum_{k=0}^{\infty} n!/k!$ is strictly between zero and one. Indeed, for $k \ge n+1$ its first term is $1/(n+1)$ and each later term is at most $1/(n+1)$ times its predecessor. Thus

$$C_n = \frac{\lfloor en! \rfloor}{(n+1)^n} \quad \text{and} \quad \frac{C_n}{\gamma_n} < e \left(\frac{n}{n+1} \right)^n. \tag{4.1}$$

For $x > 0$, the elementary logarithm inequalities $\log(1+x) > x - x^2/2$ and $\log(1+x) > x/(1+x)$ give

$$1 - n \log(1 + 1/n) < \frac{1}{2n} < \log\left(1 + \frac{1}{2(n-1)}\right). \tag{4.2}$$

For the second comparison, apply the second inequality at $x = 1/[2(n-1)]$ and observe that $x/(1+x) = 1/(2n-1) > 1/(2n)$. Taking exponentials in (4.1)–(4.2) shows

$$(n-1)\left(\frac{C_n}{\gamma_n} - 1\right) < \frac{1}{2}. \tag{4.3}$$

This estimates b_n from above. For its positivity, we proved $\sigma_s(Q) \ge 0$ for $2 \le s \le n$ in [10, equation (3.5)]. Also, by the zero-line-sum identity $\sigma_2(Y) = \|Y\|_F^2/2$, valid for every real matrix Y with zero line sums,

$$\sigma_2(Q) = \frac{\|Q\|_F^2}{2} = \frac{n-1}{2} > 0. \tag{4.4}$$

To see the identity, expand $\sigma_2(Y)$ and sum over two distinct rows and columns; after fixing one entry y_{ij}, the sum of entries in the complementary rows and columns is y_{ij} because all line sums vanish. Therefore the sum in (3.2) with Q is strictly positive, so $C_n > \gamma_n$ and $b_n > 0$.

If $n \geq 3$, (4.3) and $(n-1)^2 > n$ give

$$b_n < \frac{1}{2(n-1)^2} < \frac{1}{2n} < a_n. \tag{4.5}$$

For $n = 2$, direct substitution gives $b_2 = 1/9 < 1/4 < a_2 = 1/3$. Thus $0 < b_n < 1/(2n) < a_n$ for every $n \geq 2$.

The strict comparison makes equality analysis of the two individual estimates unnecessary: their combination separates the permanents whenever $A \neq J_n$.

4.2 Completion of the proof

Proof of Theorem 1.1. Let $n \geq 2$. Theorem 1.2 and $e^y \geq 1 + y$ give

$$\begin{aligned}\operatorname{per} A - \operatorname{per} W_n(A) &\geq \gamma_n(e^{a_n r^2} - 1 - b_n r^2) \\ &\geq \gamma_n(a_n - b_n)r^2,\end{aligned}$$

which proves (1.6). The comparison above gives $a_n > b_n$, so the last expression is strictly positive when $A \neq J_n$. At $A = J_n$, both permanents equal γ_n. For $n = 1$, the only doubly stochastic matrix is $A = J_1 = [1]$; it is fixed by W_1, and both permanents equal one. This proves (1.3) and its equality claim. $\square$

Acknowledgments

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